

MAT 213

EXAM I

Please note that you have 5 questions and 5 pages

1) (42 points) Evaluate the following integrals.

a) $\int \frac{4e^x}{e^{3x} + 2e^{2x}} dx$ Let $u = e^x$ $du = e^x dx$

$$\int \frac{4du}{u^3 + 2u^2}$$

$$\frac{A}{u^2} + \frac{B}{u} + \frac{C}{u+2} = \frac{4}{u^3 + 2u^2}$$

$$A(u+2) + Bu(u+2) + Cu^2 = 4$$

$$\begin{array}{ll} \text{for } u=2 & \boxed{C=1} \\ \text{for } u=0 & \boxed{A=2} \\ \text{for } u=1 & 2(3) + B(3) + \boxed{B=-1} = 4 \end{array}$$

$$\int \frac{4du}{u^3 + 2u^2} = \int \frac{2du}{u^2} \Rightarrow \int \frac{du}{u} + \int \frac{du}{u+2} = \frac{-2}{u} - \ln|u| + \ln|u+2| + C$$

but $u = e^x$ $\int \frac{4e^x dx}{e^{3x} + 2e^{2x}} = \frac{-2}{e^x} - \ln|e^x| + \ln|e^x + 2| + C$

14 $\int \frac{4e^x}{e^{3x} + 2e^{2x}} dx = \int \frac{4}{u^3 + 2u^2} du$

b) $\int \frac{8dx}{(4x^2 + 1)^2}$

$$\theta = \tan^{-1} 2x$$

$$2x = \tan \theta$$

$$2dx = \sec^2 \theta d\theta$$

$$4x^2 = \tan^2 \theta$$

$$4x^2 + 1 = \sec^2 \theta$$

$$\int \frac{4 \sec^2 \theta d\theta}{\sec^4 \theta} = 4 \int \frac{d\theta}{\sec^2 \theta}$$

$$= 4 \int \cos 2\theta d\theta + 2 \int d\theta$$

$$= \sin 2\theta + 2\theta + C$$

$$= 2 \sin \theta \cos \theta + 2 \tan^{-1}(2x) + C$$

$$= 2 \frac{\tan \theta}{\sec^2 \theta} + 2 \tan^{-1}(2x) + C$$

$$= 2 \frac{(2x)}{4x^2 + 1} + 2 \tan^{-1}(2x) + C$$

15 $= \frac{4x}{4x^2 + 1} + 2 \tan^{-1}(2x) + C$

$$\begin{aligned}
 & \text{c) } \int_1^{\ln x} \frac{dx}{x^2} \quad \text{let } u = \ln x \quad v = -\frac{1}{X} \\
 & \quad du = \frac{dx}{x} \quad dv = \frac{1}{X^2} dX \\
 & = \lim_{a \rightarrow \infty} \left(\frac{-\ln x}{X} \Big|_1^a + \int \frac{-dx}{X^2} \right) = \lim_{a \rightarrow \infty} \left(\frac{-\ln X}{X} \Big|_1^a - \frac{1}{X} \Big|_1^a \right) \\
 & = 0 + 0 + 0 + 1 = 1
 \end{aligned}$$

14

- 2) (10 points) Determine whether the following improper integral converges or diverges. Give reasons for your answer.

$$\lim_{a \rightarrow \infty} \int_a^{\infty} \frac{x}{x^2 \sqrt{1-\frac{1}{x^4}}} dx \quad \text{limit comparison test} \quad \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^4-1}}} = 1 \quad \frac{1}{x} \text{ diverges} \quad \text{Therefore, } \int_2^{\infty} \frac{x}{\sqrt{x^4-1}} dx \text{ diverges}$$

~~10~~

- 3) (8 points) Determine whether the following sequence $\{a_n\}, n \geq 2$ converge or diverge. Give reasons for your answer.

Sandwich theorem:

$$\frac{0}{\ln n} \leq a_n \leq \frac{2}{\ln n} \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{0}{\ln n} \leq \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} \frac{2}{\ln n}$$

$$0 \leq \lim_{n \rightarrow \infty} a_n \leq 0$$

~~8~~

$$\therefore \lim_{n \rightarrow \infty} a_n = 0 \quad \text{Converges}$$

- 4) (26 points) Test the following series for convergence and divergence. Give reasons for your answers. If a series converges, find its sum.

$$\text{a) } \sum_{n=0}^{\infty} \left(\frac{n+2}{n+1} \right)^n \quad \lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right)^n \quad \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n+2}{n+1} \right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{n+2-n-2}{(n+1)^2}}{\left(\frac{n+2}{n+1} \right) \cdot \left(-\frac{1}{n^2} \right)} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)(n+2)} = 1$$

~~10~~

$$\lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right)^n = e^1 = e \neq 0 \quad \text{Diverges}$$

$$b) \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \quad (\text{Ratio Test}) \quad \lim_{n \rightarrow \infty} \frac{(2n+2)!}{[(n+1)!]^2} \times \frac{(n!)^2}{(2n)!} = \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= 4 > 1 \quad \text{Diverges}$$

8

$$c) \sum_{n=0}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{3^n} \right] \quad \text{Difference of 2 geometric series whose sum we can find.}$$

The first geometric series has a ratio of $\frac{2}{3}$ which is < 1 . So it converges.
 The 2nd " " " " " " " " $\frac{1}{3}$ " " < 1 . So it converges.

8
 The difference of 2 series that converge converges also.
 Therefore $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$ converges

$$\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3 \quad \text{where 1st term}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n} = 3 - \frac{3}{2} = \frac{3}{2}$$

- 5) (14 points) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. Give reasons for your answers.

$$\lim_{n \rightarrow \infty} \frac{(2n+1)}{n\sqrt{4n-3}} = \lim_{n \rightarrow \infty} \frac{n(2 + \frac{1}{n})}{n^{\frac{3}{2}}\sqrt{4 - \frac{3}{n}}} = \lim_{n \rightarrow \infty} \frac{(2 + \frac{1}{n})}{\sqrt{n}\sqrt{4 - \frac{3}{n}}} = 0 \quad \left. \begin{array}{l} \text{Alternating series} \\ \text{Converges} \end{array} \right\}$$

All its terms are 6ive & it is decreasing

limit Comparison Test for $\left| \frac{(2n+1)(-1)^n}{n\sqrt{4n-3}} \right|$ to test if it converges Absolutely or Conditionally

$$\lim_{n \rightarrow \infty} \frac{(2n+1)}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{n(2 + \frac{1}{n})}{\sqrt{n^3} \sqrt{4 - \frac{3}{n}}} = \lim_{n \rightarrow \infty} \frac{(2 + \frac{1}{n})}{\sqrt{n} \sqrt{4 - \frac{3}{n}}} = 1$$

Since $\frac{1}{\sqrt{n}}$ Diverges So the series Converges Conditionally

W

**Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics**

**MAT 213
Calculus III
Exam #1
Monday November 5, 2007
Duration: 60 minutes**

Name: _____

Section: _____

Instructor: Dr. Malleh Keyrouz

Grade: 98

Directions

1. Write neatly and clearly.
2. Do not use pencils unless for graphing.
3. Only scientific calculators are allowed.

Please note that you have 5 exercises and a total number of 9 pages and that your mobile must be turned off and unseen

1) (20 points) Evaluate:

$$\text{a) } \int_{-2}^0 \frac{dx}{(x+1)^{\frac{2}{5}}} = -\int_{-2}^{-1} (x+1)^{-\frac{2}{5}} dx + \int_{-1}^0 (x+1)^{-\frac{2}{5}} dx$$

$$= \lim_{a \rightarrow -1} \int_{-2}^a (x+1)^{-\frac{2}{5}} dx + \lim_{b \rightarrow -1} \int_b^0 (x+1)^{-\frac{2}{5}} dx$$

$$= \lim_{a \rightarrow -1} \frac{5}{3} (x+1)^{\frac{3}{5}} \Big|_{-2}^a + \lim_{b \rightarrow -1} \frac{5}{3} (x+1)^{\frac{3}{5}} \Big|_b^0$$

$$= 0 + \frac{5}{3} \sqrt[3]{(-2+1)^3} + \frac{5}{3}$$

$$= \frac{5}{3} + \frac{5}{3}$$

$$= \frac{10}{3}$$



b) $\int_2^\infty \frac{dt}{t^2-1}$

$$\frac{1}{t^2-1} = \frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1}$$

$$\frac{1}{(t+1)(t-1)} = \frac{A(t-1) + B(t+1)}{(t+1)(t-1)}$$

$$\Rightarrow A = -\frac{1}{2}, \quad B = \frac{1}{2}$$

$$\begin{aligned}
 2 \int_2^\infty \frac{dt}{t^2-1} &= \lim_{a \rightarrow \infty} 2 \int_2^a \frac{-1}{2(t+1)} dt + \frac{1}{2(t-1)} dt \\
 &= \lim_{a \rightarrow \infty} \left[\frac{-1}{2(t+1)} \right]_2^a + \lim_{a \rightarrow \infty} \left[\frac{1}{2(t-1)} \right]_2^a \\
 &= \lim_{a \rightarrow \infty} -\frac{1}{2} \ln(t+1) \Big|_2^a + \lim_{a \rightarrow \infty} \frac{1}{2} \ln(t-1) \Big|_2^a \\
 &= \lim_{a \rightarrow \infty} \left(\frac{1}{2} \ln(t-1) - \frac{1}{2} \ln(t+1) \right) \Big|_2^a \\
 &= \lim_{a \rightarrow \infty} \left(\frac{1}{2} \ln\left(\frac{t-1}{t+1}\right) \right)_2^a \\
 &= \frac{1}{2} \ln 1 - \frac{1}{2} \ln \frac{1}{3} \\
 &= -\frac{1}{2} \ln \frac{1}{3} \\
 &= \frac{1}{2} \ln 3
 \end{aligned}$$

2) (16 points) Test the following integrals for convergence or divergence:

a) $\int_3^\infty \frac{4dx}{e^x + e^{-x}}$

$$e^x > 0 \quad e^{-x} > 0$$

$$e^x + e^{-x} > e^x$$

$$0 < \frac{4}{e^x + e^{-x}} < \frac{4}{e^x}$$

$$\int_3^\infty \frac{4}{e^x} dx = \lim_{a \rightarrow \infty} \int_3^a 4e^{-x} dx = \lim_{a \rightarrow \infty} -4e^{-x} \Big|_3^a = -4e^{-3} =$$

(5) converges
 $\Rightarrow \int_3^\infty \frac{4}{e^x + e^{-x}} dx$ converges by DCT

b) $\int_2^\infty \frac{2dx}{x^{\frac{7}{2}} - 1}$

Let $f(x) = \frac{2}{x^{\frac{7}{2}} - 1}$ and $g(x) = x^{-\frac{7}{2}}$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\frac{2}{x^{\frac{7}{2}} - 1}}{\frac{1}{x^{\frac{7}{2}}}} = \frac{2x^{\frac{7}{2}}}{x^{\frac{7}{2}} - 1} = 2 \in [0, \infty)$$

$f(x)$ and $g(x)$ are continuous and positive
on $[2, \infty)$ and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 2$

Hence, $\int_2^\infty f(x) dx$ and $\int_2^\infty g(x) dx$ converge together
 or diverge together.

$\int_2^\infty g(x) dx = \int_2^\infty \frac{1}{x^{\frac{7}{2}}} dx$ is a p-integral where $p > 1$
 $\Rightarrow g(x)$ converges

then $f(x)$ converges too by L.C.T

3) (24 points) Determine whether the following sequences converge or diverge. If it converges, find its limit.

a) $a_n = \left(1 + \frac{5}{n^2}\right)^n$

$\ln a_n = \ln \left(1 + \frac{5}{n^2}\right)^n = n \ln \left(1 + \frac{5}{n^2}\right) = \frac{\ln \left(1 + \frac{5}{n^2}\right)}{\frac{1}{n}}$

perform Hospital's Rule

$$\lim_{n \rightarrow \infty} \ln a_n = \lim_{n \rightarrow \infty} \frac{\frac{-10}{n^3} \left(\frac{1}{1 + \frac{5}{n^2}}\right)}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{10n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{10}{n} = 0$$

$$\ln a_n \rightarrow 0$$

$$a_n \rightarrow e^0 = 1$$

$$\lim_{n \rightarrow \infty} a_n = 1 \quad \text{converges}$$

✓

$$b) b_n = \sqrt[n^3]{n^3} \left(\frac{n+4}{n} \right)^n$$

use Rule

$$\lim_{n \rightarrow \infty} \sqrt[n^3]{n^3} = \lim_{n \rightarrow \infty} n^{3/n} \quad \ln n^{3/n} = \frac{3}{n} \ln n \rightarrow \frac{\infty}{\infty}$$

$$\text{H.R. } \lim_{n \rightarrow \infty} n^{3/n} = 3 \times 0 = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} n^{3/n} = e^0 = 1 \quad \Rightarrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left(\frac{n+4}{n} \right)^n$$
$$= \lim_{n \rightarrow \infty} \left(1 + \frac{4}{n} \right)^n$$

$$= e^4$$

| ✓

$$c) c_n = (n-1)^{\frac{1}{n-1}}$$

Let $N = n-1$ $\lim_{N \rightarrow \infty} \ln N^{\frac{1}{N}} = \lim_{N \rightarrow \infty} \frac{\ln N}{N} = 0 \rightarrow N^{\frac{1}{N}} \rightarrow e^0 = 1$

$$\lim_{n \rightarrow \infty} c_n = \lim_{N \rightarrow \infty} N^{\frac{1}{N}} = 1$$

| ✓

4) (20 points) Consider the polar coordinate curve: $r = 2\sin(2\theta)$

a) (5 points) Identify the type(s) of symmetry satisfied by the graph.

b) (5 points) Sketch the curve.

20

$$r = 2\sin(2\theta)$$

a) $(-r, -\theta)$ satisfies the equation \Rightarrow the curve is symmetric w.r.t the y-axis

$$2\sin(2\pi - 2\theta) = 2\sin(-2\theta) = -r$$

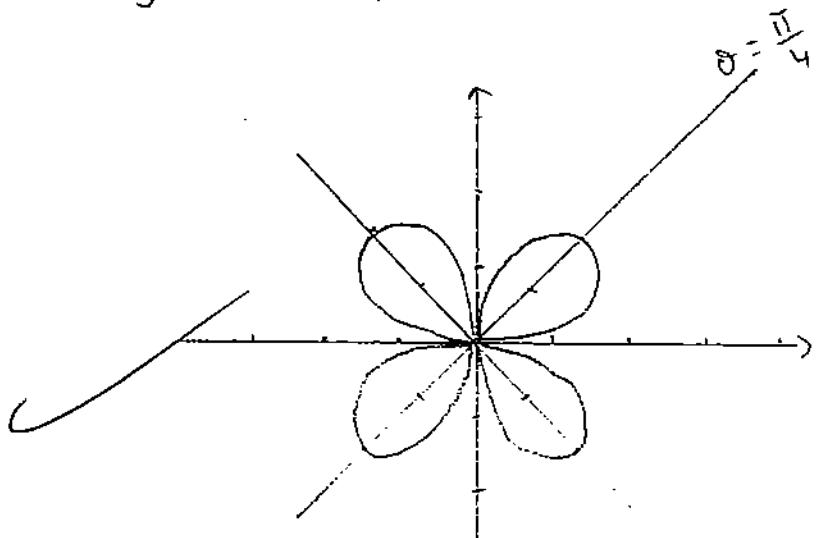
$\Rightarrow (-r, \pi - \theta)$ satisfies the equation

\Rightarrow symmetry with respect to the x-axis

Hence, the graph satisfies all symmetries about the origin, y-axis, and x-axis

b)

| θ | $r = 2\sin(2\theta)$ |
|----------|----------------------|
| 0 | 0 |
| $\pi/4$ | 2 |
| $\pi/2$ | 0 |



c) (10 points) Find the equation of the tangent to the curve at $\theta = \frac{\pi}{4}$.

$$\theta = \frac{\pi}{4} \quad r = 2 \sin\left(\frac{2\pi}{4}\right) = 2 \sin \frac{\pi}{2} = 2$$

$$(2, \frac{\pi}{4})$$

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr \sin \theta + r \cos \theta}{d\theta}}{\frac{dr \cos \theta - r \sin \theta}{d\theta}}$$

$$= \frac{4 \cos(2\theta) \sin \theta + 2 \sin(2\theta) \cos \theta}{4 \cos(2\theta) \cos \theta - 2 \sin(2\theta) \sin \theta}$$

$$= \frac{2 \cos \theta}{-2 \cos \theta} = -1$$

$$x_0 = r \cos \theta = 2 \cos \frac{\pi}{4} = \frac{2\sqrt{2}}{2} = \sqrt{2} \quad y_0 = r \sin \theta = 2 \sin \frac{\pi}{4} = \sqrt{2}$$

$$(T) : y - y_0 = m(x - x_0)$$

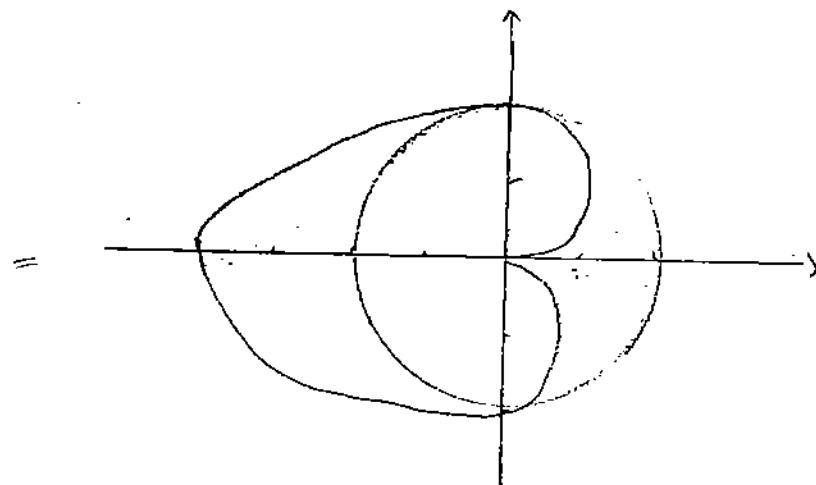
$$y - \sqrt{2} = -1(x - \sqrt{2})$$

$$\cancel{y = -x + 2\sqrt{2}}$$

- 5) (20 points) Find the area of the region shared by the circle $r = 2$ and the cardioid $r = 2 - 2\cos\theta$.

$$\begin{array}{c|c} \theta & r = 2 - 2\cos\theta \\ \hline 0 & 0 \\ \frac{\pi}{2} & 2 \\ \pi & 4 \\ \frac{3\pi}{2} & 2 \\ 2\pi & 0 \end{array}$$

20



$$\text{Area} = 2 \int_0^{\frac{\pi}{2}} \frac{(2 - 2\cos\theta)^2}{2} d\theta + 2 \int_{\frac{\pi}{2}}^{\pi} \frac{2^2}{2} d\theta$$

$$= \int_0^{\frac{\pi}{2}} (4 + 4\cos^2\theta - 8\cos\theta) d\theta + \int_{\frac{\pi}{2}}^{\pi} 4 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_0^{\frac{\pi}{2}} 4\cos^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4 \cos 2\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_0^{\frac{\pi}{2}} 4\cos^2\theta d\theta - 8 \int_0^{\frac{\pi}{2}} \cos\theta d\theta + \int_{\frac{\pi}{2}}^{\pi} 4 d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4 d\theta + 2 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta - 8 \int_0^{\frac{\pi}{2}} \cos\theta d\theta + \int_{\frac{\pi}{2}}^{\pi} 4 d\theta$$

$$= [4\theta]_0^{\frac{\pi}{2}} + [2\theta]_0^{\frac{\pi}{2}} + [\sin 2\theta]_0^{\frac{\pi}{2}} - 8 [\sin\theta]_0^{\frac{\pi}{2}} + [4\theta]_{\frac{\pi}{2}}^{\pi}$$

$$= 2\pi + \pi - 8 + 4\pi - 2\pi$$

$$= 5\pi - 8$$

**Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics**

**MAT 213
Calculus III
Exam #2**

**Monday December 17, 2007
Duration: 60 minutes**

Name: _____

Section: _____

Instructor: Dr. Mahab Keyrouz

Grade: _____

Directions

Scientific calculators are allowed

Please note that you have 5 exercises and a total number of 7 pages and that your mobile must be turned off and unseen

- 1) (14 points) Say whether the following statements are TRUE or FALSE. When FALSE, give a counterexample:

a) Given that $0 < a_n \leq b_n$ and that $\sum_0^{\infty} b_n$ diverges, then $\sum_0^{\infty} a_n$ diverges.

14 False

counter example:

$$n^2 > n > 0$$

$$0 < \frac{1}{n^2} < \frac{1}{n} \quad \text{let } a_n = \frac{1}{n^2} \quad b_n =$$

although $\frac{1}{n^2} < \frac{1}{n}$

$\sum_0^{\infty} \frac{1}{n}$ diverges (harmonic series)

$\sum_0^{\infty} \frac{1}{n^2}$ converges (p-series $p > 1$)

b) If $\sum_0^{\infty} a_n$ converges to A and $\sum_0^{\infty} b_n$ converges to B , then $\sum_0^{\infty} a_n b_n$ converges to AB

False ✓

counter example: $a_n = \left(\frac{1}{2}\right)^n \quad b_n = \left(\frac{2}{3}\right)^n$

$$\sum_0^{\infty} a_n = \sum_0^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2 = A$$

$$\sum_0^{\infty} b_n = \sum_0^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3 = B$$

$$\sum_0^{\infty} a_n b_n = \sum_0^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \neq AB = 6$$

2) (40 points) Test the following series for convergence or divergence. Justify your answer.

a) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ $a_n = \frac{n!}{(2n+1)!} > 0$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)n!}{n!} \cdot \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{(2n+3)(2n+2)} \\ &= 0 < 1 \\ \therefore \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!} &\text{ converges by ratio test} \end{aligned}$$

20

b) $\sum_{n=1}^{\infty} \left(\frac{n-4}{n}\right)^n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{n-4}{n}$$

$$= \sum_{n=1}^{\infty} \left(1 - \frac{4}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{4}{n}\right)^n = e^{-4} \neq 0$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{n-4}{n}\right)^n \text{ diverges by nth term test}$$

$$c) \sum_{n=1}^{\infty} n^2 e^{-n}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n} \quad a_n = \frac{n^2}{e^n} > 0$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^2}{e^n}} = \lim_{n \rightarrow \infty} \frac{n^2/e}{e} = \frac{1}{e} < 1$$

| 0

$\therefore \sum_{n=1}^{\infty} n^2 e^{-n}$ converges by
nth root test

$$d) \sum_{n=1}^{\infty} \frac{\sin n + 2}{3^n}$$

$$-1 < \sin n < 1$$

$$1 < \sin n + 2 < 3$$

$$\frac{1}{3^n} < \frac{\sin n + 2}{3^n} < \frac{3}{3^n}$$

$$0 < \frac{\sin n + 2}{3^n} < \left(\frac{1}{3}\right)^{n-1}$$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$ converges (geometric series with $r = \frac{1}{3} < 1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin n + 2}{3^n}$ converges by direct comparison test.

3) (12 points) Determine whether the following series converges absolutely, converges conditionally, or diverges. Give reasons for your answer. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n \ln n}{(n+1) \ln(n+1)} \right| \quad \left(\frac{1}{n \ln n} \right)' = \frac{-(\ln n + 1)}{(n \ln n)^2} < 0$$

$$b_n = |a_n| = \frac{1}{n \ln n}$$

b_n is positive, decreasing, and continuous on $(2, \infty)$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ and $\int_2^{\infty} \frac{dx}{x \ln x}$ both converge or diverge together

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln x} &= \left[\ln(\ln x) \right]_2^{\infty} = \infty \quad \text{so} \\ &\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges} \end{aligned}$$

Consider $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ $b_n = \frac{1}{n \ln n}$

① $b_n > 0$

② $b_{n+1} \leq b_n$ for $n \geq 2$

③ $b_n \rightarrow 0$

$\Rightarrow \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ converges by alternating series test

in particular $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ converges conditionally

4) (18 points) Consider the power series: $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n2^n}$

Find the series radius and interval of convergence. For what values of x does the series converge absolutely, conditionally or diverge?

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n2^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x+3)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x+3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x+3}{2} \right| = \rho \end{aligned}$$

$$\rho < 1$$

$$\left| \frac{x+3}{2} \right| < 1$$

$$-2 < x+3 < 2$$

$$\underline{-5 < x < -1}$$

$$-5 < x < -1$$

$$R = 2$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges conditionally

(alternating harmonic series)

at $x = -1$

$$\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (harmonic series)

interval of convergence: ~~$[-5, -1]$~~ $[-5, -1[$

interval of absolute convergence: $[-5, -1[$

interval of divergence: $(-\infty, -5)$

it diverges elsewhere

5) (16 points) Find the first four nonzero terms for $f(x) = e^x \cos x$ by multiplying together the MacLaurin series for e^x and for $\cos x$.

MacLaurin series for e^x :

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

MacLaurin series for $\cos x$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$e^x \cos x: \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots \right)$$

$$= 1 + x - \frac{x^2}{2} + \frac{x^2}{2} + \left(\frac{-1}{2} - \frac{1}{2 \cdot 3!} \right) x^3 + \left(\frac{1}{4!} - \frac{1}{4 \cdot 4!} \right) x^4 + \left(\frac{1}{4!} - \frac{1}{2 \cdot 3!} \right) x^5 \dots$$

$$= 1 + x + \left(-\frac{3+1}{6} \right) x^3 + \left(\frac{1}{4!} - \frac{3!}{4!} + \frac{1}{4!} \right) x^4 \dots$$

~~$$= 1 + \left(-\frac{x^3}{3!} + \left(\frac{1}{4!} - \frac{1}{4 \cdot 4!} \right) x^4 + \left(\frac{1}{4!} + \frac{1}{2 \cdot 3!} \right) x^5 \right) \dots$$~~

~~$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{24} \dots$$~~

✓