1) Find the inverse of $A = \begin{pmatrix} 1 & 3 & 0 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. (10 points)

$$\boxed{\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right)} \xrightarrow[R_4/2 \to R_4]{R_3/2 \to R_3} \left(\begin{array}{cccc|cccc} 1 & 3 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$R_4 + R_3 \to R_3$
$3R_4 + R_2 \to R_2$
$3R_4 + R_1 \to R_1$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 1 & 0 & 0 & 3/2 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right) \quad -2R_3 + R_2 \to R_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right) \quad -3R_2 + R_1 \to R_1$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

2) Let V be the vector space of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let W be the subset of V consisting of all 2×2 matrices such that $a-3b=0$ and $2c+d=0$. Determine whether W is a subspace of V or not. If W is a subspace determine a basis and the dimension of W, otherwise (if it is not a subspace) skip this part. **(10 points)**

* $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a-3b=0 \Rightarrow a=3b \\ 2c+d=0 \Rightarrow d=-2c \end{matrix} \right\}$

$W = \left\{ \begin{pmatrix} 3b & b \\ c & -2c \end{pmatrix} ; b, c \in \mathbb{R} \right\}$

let $M_1 = \begin{pmatrix} 3b_1 & b_1 \\ c_1 & -2c_1 \end{pmatrix} \in W$ and $M_2 = \begin{pmatrix} 3b_2 & b_2 \\ c_2 & -2c_2 \end{pmatrix} \in W$

$M_1 + M_2 = \begin{pmatrix} 3(b_1+b_2) & b_1+b_2 \\ c_1+c_2 & -2(c_1+c_2) \end{pmatrix} \in W.$

let $k \in \mathbb{R}$

$k.M_1 = \begin{pmatrix} 3kb_1 & kb_1 \\ kc_1 & -2kc_1 \end{pmatrix} \in W$

$\Rightarrow$ W is a sobspace of V

* $\begin{pmatrix} 3b & b \\ c & -2c \end{pmatrix} = \begin{pmatrix} 3b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & -2c \end{pmatrix}$

$= b\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$

Basis $= \left\{ \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} \right\}$

$\dim(W) = 2.$

3) Solve the following system of equations by Gauss-Jordan elimination:

$$x_1 + x_2 - 2x_3 - 2x_4 - 2x_5 = 2$$
$$3x_1 + 2x_2 - 3x_3 - 2x_4 - 8x_5 = 1. \qquad \textbf{(13 points)}$$
$$x_1 + x_3 - 4x_4 + 2x_5 = 15$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & -2 & -2 & 2 \\ 3 & 2 & -3 & -2 & -8 & 1 \\ 1 & 0 & 1 & -4 & 2 & 15 \end{array}\right) \begin{array}{l} -3R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & -2 & -2 & 2 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 0 & -1 & 3 & -2 & 4 & 13 \end{array}\right) \begin{array}{l} R_2 + R_1 \to R_1 \\ -R_2 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & -4 & -3 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 0 & 0 & 0 & -6 & 6 & 18 \end{array}\right) \begin{array}{l} -R_2 \to R_2 \\ -\frac{1}{6}R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & -4 & 2 & 5 \\ 0 & 0 & 0 & 1 & -1 & -3 \end{array}\right) \begin{array}{l} 4R_3 + R_2 \to R_2 \\ -2R_3 + R_1 \to R_1 \end{array}$$

$$\boxed{\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -7 \\ 0 & 0 & 0 & 1 & -1 & -3 \end{array}\right)}$$

x_3 and x_5 are the free variables

$$x_4 - x_5 = -3 \Rightarrow \boxed{x_4 = x_5 - 3}$$

$$x_2 - 3x_3 - 2x_5 = -7$$
$$\boxed{x_2 = 3x_3 + 2x_5 - 7}$$

$$x_1 + x_3 - 2x_5 = 3$$
$$\boxed{x_1 = -x_3 + 2x_5 + 3}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 + 2x_5 + 3 \\ 3x_3 + 2x_5 - 7 \\ x_3 \\ x_5 - 3 \\ x_5 \end{pmatrix} \qquad \begin{array}{l} x_3 \in \mathbb{R} \\ x_5 \in \mathbb{R} \end{array}$$

4) Let $V = R^2$ with the operations: $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1 - 1,\ a_2 + b_2 - 2)$, and $k(a_1, a_2) = (ka_1 - k + 1, ka_2 - 2k + 2)$. Show that V with these two operations satisfy the first **six** axioms for a vector space. **(10 points)**

① let $\vec{u} = (a_1, a_2) \in V$, $\vec{v} = (b_1, b_2) \in V$

$\boxed{\vec{u} + \vec{v}} \overset{?}{=} (a_1 + b_1 - 1,\ a_2 + b_2 - 2) \in V.$

② $\vec{u} + \vec{v} = (a_1 + b_1 - 1,\ a_2 + b_2 - 2)$ $\Rightarrow \vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\vec{v} + \vec{u} = (b_1 + a_1 - 1,\ b_2 + a_2 - 2) = (a_1 + b_1 - 1,\ a_2 + b_2 - 2)$

③ $(\vec{u} + \vec{v}) + \vec{w} = (a_1 + b_1 - 1,\ a_2 + b_2 - 2) + (c_1, c_2)$

$= (a_1 + b_1 + c_1 - 2,\ a_2 + b_2 + c_2 - 4)$

$\vec{u} + (\vec{v} + \vec{w}) = (a_1, a_2) + (b_1 + c_1 - 1,\ b_2 + c_2 - 2)$

$= (a_1 + b_1 + c_1 - 2,\ a_2 + b_2 + c_2 - 4)$

$\Rightarrow (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ ✓

④ $\vec{0} + \vec{u} = \vec{u}$

① $(d_1, d_2) + (a_1, a_2) = (a_1, a_2)$

$(d_1 + a_1 - 1,\ d_2 + a_2 - 2) = (a_1, a_2)$

$d_1 + a_1 - 1 = a_1 \Rightarrow \boxed{d_1 = 1}$

$d_2 + a_2 - 2 = a_2 \Rightarrow \boxed{d_2 = 2}$

$\Rightarrow \boxed{\vec{0} = (1, 2)}$

⑤ $-\vec{u} = -1(a_1, a_2) = (-a_1 + 1 + 1,\ -a_2 + 2 + 2)$

$= (-a_1 + 2,\ -a_2 + 4)$

$\vec{u} + (-\vec{u}) \overset{??}{=} \vec{0}$ $(a_1, a_2) + (-a_1 + 2,\ -a_2 + 4) = (a_1 - a_1 + 2 - 1,\ a_2 - a_2 + 4 - 2)$

$= (1, 2) = \vec{0}$ ✓

⑥ $k \cdot \vec{u} = (ka_1 - k + 1,\ ka_2 - 2k + 2) \in V$

5) a) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 7 & 0 \\ 1 & 8 & 1 \end{pmatrix}.$$ (8 points)

A is lower triangular $\Rightarrow$ $\boxed{\lambda_1 = -1}$, $\boxed{\lambda_2 = 7}$, $\boxed{\lambda_3 = 1}$

let $\vec{v}_1$ be the e$\vec{v}$ corresp. to $\lambda_1 = -1$

$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 1 & 8 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 8 & 2 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad -2R_1 + R_2 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & 8 & 2 & 0 \\ 0 & -8 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-8x_2 - 4x_3 = 0$

$\boxed{x_2 = -\frac{1}{2}x_3}$

$x_1 + 8x_2 + 2x_3 = 0$

$x_1 - 4x_3 + 2x_3 = 0$

$\boxed{x_1 = 2x_3}$

$\lambda_1 = -1$ $\quad \vec{v}_1 = \begin{pmatrix} 2t \\ -\frac{1}{2}t \\ t \end{pmatrix}$; $t \neq 0$

$\longrightarrow$ any vector of this form is an eigenvector of A corresp. to $\lambda_1 = -1$

let $\vec{v}_2$ be e$\vec{v}$ corresp to $\lambda_2 = 7$

$(A - 7I)\vec{v}_2 = \vec{0}$ $\quad \left(\begin{array}{ccc|c} -8 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 8 & -6 & 0 \end{array}\right)$

$-8x_1 = 0 \Rightarrow \boxed{x_1 = 0}$

$\cancel{x_1}^{0} + 8x_2 - 6x_3 = 0 \Rightarrow \boxed{x_2 = \frac{3}{4}x_3}$

$\lambda_2 = 7$ $\quad \vec{v}_2 = \begin{pmatrix} 0 \\ \frac{3}{4}t \\ t \end{pmatrix}$; $t \neq 0$.

let $\vec{v}_3$ be e$\vec{v}$ corresp to $\lambda_3 = 1$

$(A - I)\vec{v}_3 = \vec{0}$ $\quad \left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 1 & 8 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1 + R_2 \to R_2 \\ \frac{1}{2}R_1 + R_3 \to R_3 \end{array} \quad \left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{array}\right)$

$x_1 = 0$, $x_2 = 0$

$\lambda_3 = 1$ $\quad \boxed{\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}}$; $t \in \mathbb{R}$, $t \neq 0$.

b) Find a matrix P that diagonalizes A, and determine the diagonal form of A, and evaluate A^n for any positive integer n. **(6 points)**

let $t = 1$ in $\vec{v_1}$, $\vec{v_2}$ & $\vec{v_3}$

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ P = & & & \end{array}\begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & 3/4 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} A P = D \iff A = P D P^{-1}$$

$$A^n = P D^n P^{-1}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 3/4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{4}R_1 + R_2 \to R_2 \\ -\frac{1}{2}R_1 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3/4 & 0 & \frac{1}{4} & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} 4/3\, R_2 \to R_2 \\ \frac{1}{2}R_1 \to R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1/3 & 4/3 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

$$-R_2 + R_3 \to R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1/3 & 4/3 & 0 \\ 0 & 0 & 1 & -\frac{5}{6} & -4/3 & 1 \end{array}\right) \Rightarrow \boxed{P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & 0 \\ -\frac{5}{6} & -\frac{4}{3} & 1 \end{pmatrix}}$$

$$A^n = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{4} & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 7^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & 0 \\ -\frac{5}{6} & -\frac{4}{3} & 1 \end{pmatrix}$$

6) Let $v_1 = (2, -1, 0)$, $v_2 = (-1, 0, 1)$, $v_3 = (0, 1, 2)$, $v_4 = (-3, 2, 1)$ be 4 vectors in $\mathbf{R}^3$. Show that v_1, v_2, v_3 form a basis for $\mathbf{R}^3$, and write v_4 as a linear combination of v_1, v_2, v_3. **(8 points)**

determinant $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(0-1) + 1(-2-0) = -2-2 = -4 \boxed{\neq 0}$

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent set

number of vectors $= 3 = \dim(\mathbb{R}^3)$

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis for $\mathbb{R}^3$.

$\Downarrow$

$\boxed{\text{BASIS}}$

$\vec{v_4} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \Rightarrow \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_1 \\ -c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -c_2 \\ 0 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_3 \\ 2c_3 \end{pmatrix}$

$\left(\begin{array}{ccc|c} 2 & -1 & 0 & -3 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right) \Rightarrow$ $c_1 = -\frac{3}{2}$, $c_2 = 0$, $c_3 = \frac{1}{2}$

$\boxed{\vec{v_4} = -\frac{3}{2}\vec{v_1} + \frac{1}{2}\vec{v_3}}$

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7) a) Show that if a matrix A is row equivalent to a matrix B (i.e. one can obtain B from A by elementary row operations), then there exists an invertible matrix Q such that $QA = B$ (Hint: use elementary matrices) (5 points)

If A is row equivalent to B then there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

but $E_1, E_2, \ldots, E_k$ are elementary $\Rightarrow$ invertible

$\Rightarrow E_k \cdot E_{k-1} \cdots E_2 \cdot E_1$ is invertible (product of invertible matrices is invertible)

take $Q = E_k E_{k-1} \cdots E_2 \cdot E_1$.

b) Show that if A can be written as the product of elementary matrices, then the linear system $Ax = 0$ has only the trivial solution. (5 points)

If A can be written as product of elementary matrices then A is invertible $\Rightarrow A\vec{x} = \vec{0}$ has a unique solution which is the trivial solution.

$$A\vec{x} = \vec{0}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{0}$$

$$\vec{x} = \vec{0}.$$

8) Prove the following:

a) If λ is **not** an eigenvalue for a matrix A, then the homogeneous system $B\vec{x} = 0$, where $B = (A - \lambda I)^2$, has only the trivial solution $\vec{x} = 0$. **(5 points)**

If λ is not an e.v $\Rightarrow \det(A - \lambda I) \neq 0$.

$\det(B) = \det((A-\lambda I)^2) = \underbrace{\det(A-\lambda I)}_{\neq 0} \cdot \underbrace{\det(A-\lambda I)}_{\neq 0}$

$\Rightarrow \det(B) \neq 0$

hence B is invertible

$\Rightarrow B\vec{x} = \vec{0}$ has only the trivial solution.

b) The inverse of an invertible symmetric matrix, is itself symmetric. **(5 points)**

A is symmetric $\Leftrightarrow A^t = A$

$AA^{-1} = I$

$(AA^{-1})^t = I^t \Leftrightarrow (A^{-1})^t A^t = I \Leftrightarrow (A^{-1})^t A = I$

$\Leftrightarrow (A^{-1})^t A A^{-1} = I A^{-1} \Leftrightarrow (A^{-1})^t = A^{-1}$

$\Leftrightarrow A^{-1}$ is symmetric.

c) The product of two invertible matrices (of same size) is invertible. **(5 points)**

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)(B^{-1}A^{-1}) = AIA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}IB = I$

9) Use Cramer's rule to solve the linear system (10 points)

$$x + 2y + z = 5$$
$$2x + 2y + z = 6$$
$$x + 2y + 3z = 9$$

$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{pmatrix}$

$x = \dfrac{\det(A_1)}{\det(A)}$, $y = \dfrac{\det(A_2)}{\det(A)}$, $z = \dfrac{\det(A_3)}{\det(A)}$

$\det(A) = 1(6-2) - 2(6-1) + 1(4-2) = 4 - 10 + 2 = -4$

$\det(A_1) = 5(6-2) - 2(18-9) + 1(12-18) = 20 - 18 - 6 = -4$

$\det(A_2) = 1(18-9) - 5(6-1) + 1(18-6) = 9 - 25 + 12 = -4$

$\det(A_3) = 1(18-12) - 2(18-6) + 5(4-2) = 6 - 24 + 10 = -8$

$\boxed{x = \frac{-4}{-4} = 1}$ $\boxed{y = 1}$ $\boxed{z = 2}$