1) **(14 points)** Evaluate the determinant of the matrix: $A = \begin{bmatrix} 1 & 0 & 4 & 1 & 7 \\ 3 & 5 & 4 & 1 & 7 \\ 2 & 0 & 8 & 2 & 9 \\ 3 & 0 & 8 & 3 & 5 \\ 4 & 2 & 3 & 4 & 1 \end{bmatrix}$.

$$\det(A) = 5 \times \begin{vmatrix} 1 & 4 & 1 & 7 \\ 2 & 8 & 2 & 9 \\ 3 & 8 & 3 & 5 \\ 4 & 3 & 4 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 4 & 1 & 7 \\ 3 & 4 & 1 & 7 \\ 2 & 8 & 2 & 9 \\ 3 & 8 & 3 & 5 \end{vmatrix} \begin{matrix} \\ (-3) \\ (-2) \\ (-3) \end{matrix}$$

0 since 2 proportional column

$$\det(A) = -2 \times \begin{vmatrix} 1 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \\ 0 & 0 & 0 & -5 \\ 0 & -4 & 0 & -16 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \\ 0 & -4 & 0 & -16 \\ 0 & 0 & 0 & -5 \end{vmatrix} (-1/2)$$

$$= 2 \times \begin{vmatrix} 1 & 4 & 1 & 7 \\ 0 & -8 & -2 & -14 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & -5 \end{vmatrix} = 2 \times 1 \times (-8) \times (1) \times (-5)$$

$$= 80$$

14

2) (16 points) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Given that $\det(A) = 5$, find each of the following:

a) **(4 points)** $\det(4A^{-1})$

$= 4^3 \det(A^{-1}) = \dfrac{4^3 \cancel{\det}(A)}{\det(A)} = 4^3 \times 5 = 320.$

-2

14

b) **(6 points)** $\begin{vmatrix} d & e & f \\ 4a & 4b & 4c \\ g+2d & h+2e & i+2f \end{vmatrix} \xrightarrow{R_3 + (-2)R_1} = \begin{vmatrix} d & e & f \\ 4a & 4b & 4c \\ g & h & i \end{vmatrix}$

$= 4 \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -4 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4 \det(A) = -20.$

c) **(6 points)** $\begin{vmatrix} a+2d & 2a+d & g \\ b+2e & 2b+e & h \\ c+2f & 2c+f & i \end{vmatrix}$ multiply the first column by -2 and add to the 2nd.

$= \begin{vmatrix} a+2d & -3d & g \\ b+2e & -3e & h \\ c+2f & -3f & i \end{vmatrix} = -3 \begin{vmatrix} a+2d & d & g \\ b+2e & e & h \\ c+2f & f & i \end{vmatrix} \quad C_1 + (-2)C_2$

$= -3 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = -3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= -15.$

3) **(14 points)** For the matrix $A=\begin{bmatrix}1 & 2 & 3\\ -2 & -1 & 0\\ 5 & 4 & 2\end{bmatrix}$, find:

a) **(10 points)** $adj(A)$

$C_{11} = + \begin{vmatrix}-1 & 0\\ 4 & 2\end{vmatrix} = -2$

$C_{12} = - \begin{vmatrix}-2 & 0\\ 5 & 2\end{vmatrix} = 4$

$C_{13} = + \begin{vmatrix}-2 & -1\\ 5 & 4\end{vmatrix} = -3$

$C_{21} = - \begin{vmatrix}2 & 3\\ 4 & 2\end{vmatrix} = 8$

$C_{22} = \begin{vmatrix}1 & 3\\ 5 & 2\end{vmatrix} = -13$

$C_{23} = - \begin{vmatrix}1 & 2\\ 5 & 4\end{vmatrix} = 6$

$C_{31} = + \begin{vmatrix}2 & 3\\ -1 & 0\end{vmatrix} = 3$

$C_{32} = - \begin{vmatrix}1 & 3\\ -2 & 0\end{vmatrix} = -6$

$C_{33} = \begin{vmatrix}1 & 2\\ -2 & -1\end{vmatrix} = 3$

$C = \begin{bmatrix}-2 & 4 & -3\\ 8 & -13 & 6\\ 3 & -6 & 3\end{bmatrix}$

$adj(A) = C^T = \begin{bmatrix}-2 & 8 & 3\\ 4 & -13 & -6\\ -3 & 6 & 3\end{bmatrix}$

b) **(4 points)** Use $adj(A)$ to find A^{-1}.

~~$A^{-1} = adj(A)$~~

$\det(A) = \begin{vmatrix}1 & 2 & 3\\ -2 & -1 & 0\\ 5 & 4 & 2\end{vmatrix}$

$= 3\begin{vmatrix}-2 & -1\\ 5 & 4\end{vmatrix} + 2\begin{vmatrix}1 & 2\\ -2 & -1\end{vmatrix} = -9 + 6 = -3 \neq 0$

So A invertible

$A^{-1} = \dfrac{adj(A)}{\det(A)} = \begin{bmatrix}+2/3 & -8/3 & -1\\ -4/3 & +13/3 & +2\\ 1 & -2 & -1\end{bmatrix}$

14

4) (12 points)

a) **(6 points)** Use determinants to find the values of k for which the following homogeneous system will have non-trivial solutions:

$$(k-5)x-2y=0$$
$$x+(k-2)y=0$$

$A = \begin{bmatrix} K-5 & -2 \\ 1 & K-2 \end{bmatrix}$

non trivial solution so A not invertible so det (A) = 0

$\det(A) = (K-5)(K-2) - (1)(-2) = K^2 - 7K + 10 + 2$

$= K^2 - 7K + 12 = 0$ $\quad \Delta = 49 - 48 = 1.$

$K = \frac{7 \pm \sqrt{1}}{2} \Rightarrow K_1 = 3 \;;\; K_2 = 4.$

6

b) **(6 points)** Use Cramer's rule to solve the following system for z only:

$$x+2y+3z=1$$
$$y+2z=2$$
$$3x-z=3$$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{vmatrix} = 1\times(-1) - 2\times(-6) + 3(-3)$

$= -1 + 12 - 9 = 2 \neq 0$ so A is invertible we can use gramer rule

$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ $\quad \det(A_3) = 3 - 2\times(-6) - 3 = 12$

$z = \frac{\det(A_3)}{\det(A)} = \frac{12}{2} = 6.$

6

12

5) **(20 points)** Given that A, B and C are $n \times n$ matrices. Prove each of the following statements:

a) **(4 points)** Suppose B is obtained from A by multiplying the i^{th} row of A by k. Use the definition of determinants to show that $\det(B) = k\det(A)$.

Since The determinant is The summation of all signed elementary product ~~Then~~ and each elementary product contain an entry from each row and column The The summation will be multiplied by K if one row or column is multiplied by K.

det (A) = $\sum \pm a_{1j_1} a_{2j_2} \dots a_{nj_n}$ If The ith mult by K. Then

det (B) = $\sum \pm a_{1j_1} a_{2j_2} \dots K a_{ij_i} \dots$

$= K \sum \pm a_{1j_1} a_{2j_2} \dots a_{nj_n}$

= K det (A)

4

b) **(4 points)** If $\det(A) \neq 0$ and $AB = AC$, then $B = C$.

if det (A) ≠ 0 So A is invertible Then

If AB = AC Then $\underbrace{A^{-1}A}_{I} B = \underbrace{A^{-1}A}_{I} C$

B = C

4

c) **(6 points)** $\det(AB) = \det(A) \cdot \det(B)$.

* ① If A is not invertible Then det (A) = 0, and AB not invertible So det (AB) = 0 = 0 × det (B) = det (A) det (B)

* If A is invertible The $A = E_1 E_2 E_3 \dots E_r$

det (AB) = det ($E_1 E_2 \dots E_r B$) = det (E_1) det (E_2) ... det (E_r) × det (B)

= det ($E_1 E_2 \dots E_r$) × det (B) = det (A) × det (B).

6

So from ① and ②

det (AB) = det (A) × det (B)

for every A.

d) **(6 points)** Let A be an $n \times n$ matrix with n odd. Prove that if A is skew-symmetric, then $\det(A) = 0$. (Recall: A is skew-symmetric if $A^T = -A$).

20

$A^T = -A$. $\det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$ n is odd.

but $\det(A^T) = \det(A)$.

So $\det(A^T) = \det(A) = -\det(A)$.

So ~~ide~~ $\det(A) + \det(A) = 0$

$2\det(A) = 0$.

$\boxed{\det(A) = 0}$

6

6) **(8 points)** Let A be an $n \times n$ matrix which is not invertible.

a) **(4 points)** Show that $A\,adj(A) = O_{n \times n}$.

~~$A\,adj(A)$~~ $= \begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix}$

$A\,adj(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & & & \\ c_{13} & & c_{33} & & \\ \vdots & & & & \\ c_{1n} & & & & c_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & 0 & \cdots & 0 \\ 0 & 0 & \det(A) & \cdots & 0 \\ 0 & & \cdots & & \det(A) \end{bmatrix}$

each entry is ~~…~~ $a_{i1}c_{j1} + a_{i2}c_{j2} \ldots a_{in}c_{jn} = \det A$ if $i = j$ and 0 if $i \neq j$.

But A is not invertible so $\det(A) = 0$

Then $A\,adj(A) = \cancel{0\det}(A)I_{n \times n} = 0I_{n \times n} = O_{n \times n}$

4

b) **(4 points)** Use part (a) to deduce that $adj(A)$ is not invertible.

$A\,adj(A) = O_{n \times n}$.

~~$\det(A\,adj(A)) = \det(O_{n \times n}) = 0 = \det(A)\det(A\,adj(A))$~~.

~~$\det(A\,adj(A)) = \det(A)I$~~

2

$\det(A\,adj(A)) = \det(O_n) = 0$

$\det(A)\det(A\,adj(A)) = 0$

$\det(A\,adj(A)) = 0$

So?

$A\,adj(A)$ ~~not~~ invertible

6

7) **(16 points)** Let $V=\mathbb{R}^2$, the set of all pairs of real numbers, with the operations:

$$(x,y)+(x',y')=(xx',2yy')$$
$$k(x,y)=(kx,ky)$$

Check all 10 axioms that must hold for V to be a vector space, and determine which axioms are true and which are false. Then determine whether or not V is a vector space.

let $\vec{u}=(x,y)$ $\vec{v}=(x',y')$ and $\vec{w}=(x'',y'')$ in V

axiom 1: $\vec{u}+\vec{v}=(x,y)+(x',y')=(xx',2yy')$ pair of real # Then $\vec{u}+\vec{v}$ is in V true

axiom 2: $\vec{u}+\vec{v}=(xx',2yy')$. $\vec{v}+\vec{u}=(x',y')+(x,y)=(x'x,2y'y)$

So $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ true

axiom 3: $\vec{u}+(\vec{v}+\vec{w})=(x,y)+(x'x'',2y'y'')=(xx'x'',4yy'y'')$

$(\vec{u}+\vec{v})+\vec{w}=(xx',2yy')+(x'',y'')=(xx'x'',4yy'y'')$

$\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ So true

axiom 4: $\vec{0}=(1,\frac{1}{2})$ So $\vec{u}+\vec{0}=(x,y)+(1,\frac{1}{2})=(x,y)$

and $\vec{0}+\vec{u}=(1,\frac{1}{2})+(x,y)=(x,y)$

Then $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$ true

axiom 5: $-\vec{u}=(\frac{1}{x},\frac{1}{4y})$ in V $\vec{u}+(-\vec{u})=(x,y)+(\frac{1}{x},\frac{1}{4y})=(1,\frac{1}{2})$

but $-\vec{u}$ not in V $-\vec{u}+\vec{u}=(\frac{1}{x},\frac{1}{4y})+(x,y)=(1,\frac{1}{2})=\vec{0}$

Also (or any $(x,0)$ any $(0,y)$) no inverse ~~for~~ $(0,0)$ $-\vec{u}$ not defining so false

bonus +2

axiom 6 $k\vec{u}=(kx,ky)$ in V So true

axiom 7: $k(\vec{u}+\vec{v})=k(xx',2yy')=(kxx',2kyy')$

$k\vec{u}+k\vec{v}=(kx,ky)+(kx',ky')=(k^2xx',2k^2yy')\neq k(\vec{u}+\vec{v})$

axiom 7 fails.