MAT 215 – Linear Algebra I
Spring 2001 – Final Exam
Duration: 2 hours

1) Let $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & a \\ 1 & a & 0 \end{pmatrix}$, and $b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Give conditions on a so that $Ax = b$ has:
   a) A unique solution
   b) More than one solution
   c) No solution

(15 points)

2) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix}$.

   a) Compute A^2, A^3 and show that $A^3 = 9A - 8I$.
   b) Deduce that A is invertible and express A^{-1} in terms of A^2 and I.
   c) Find A^{-1}.

(15 points)

3) Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & +1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

   a) Prove that A is row equivalent to B.
   b) Find elementary matrices E_1, E_2, E_3 so that $B = E_3 E_2 E_1 A$.

(10 points)

4) Use Gaussian elimination to compute the determinant of

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

(10 points)

5) Let $W = \left\{ (x,y) \in R^2 \ / \ x.y \geq 0 \right\}$ be a subset of R^2.

   a) Prove that W is closed under scalar multiplication
   b) Find a specific example to show that W is not a subspace of V.

(10 points)

6) Let $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & 2 & -2 & 1 \\ 3 & -3 & 3 & 2 \end{pmatrix}$.

a) Find the rank of A and deduce that the subset
$S = \{v_1 = (1, -1, 1, 1), v_2 = (2, 1, -1, 3), v_3 = (4, 2, -2, 1), v_4 = (3, -3, 3, 2)\}$
is linearly dependant.

b) Find a basis B for $\text{lin}(S)$ such that B is a subset of S.

(10 points)

7) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

a) Find the eigenvalues and eigenvectors of A.
b) Is A diagonalizable? Why?
c) Find an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.
d) Calculate $\det(A^{10})$ and $tr(A^{10})$.

(20 points)

8) Let A be an invertible $n \times n$ matrix and λ be an eigenvalue of A. show that λ^{-1} is an eigenvalue of A^{-1}.

(10 points)