1) By using the **determinant** of the coefficient matrix of the following homogeneous system, find a relation between α and β so that the system has nontrivial solutions:

$$x+y+\alpha z=0$$
$$x+y+\beta z=0 \qquad \text{(20 pts)}$$
$$\alpha x+\beta y+z=0$$

$AX=0$ if $|A|=0 \Rightarrow$ system has non trival Solution (infinitely Many)

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$$A=\begin{pmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{pmatrix}$$

$$|A|=0 \Rightarrow \begin{vmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{vmatrix}=0 \Rightarrow$$

$$1\begin{vmatrix} 1 & \beta \\ \beta & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & \beta \\ \alpha & 1 \end{vmatrix} + \alpha\begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix}=0$$

$$1-\beta^2-1[1-\alpha\beta]+\alpha[\beta-\alpha]=0$$

$$\cancel{1}-\beta^2-\cancel{1}+\alpha\beta+\alpha\beta-\alpha^2=0$$

$$-\alpha^2-\beta^2+2\alpha\beta=0$$

$$-(\alpha-\beta)^2=0$$

then $\alpha=\beta$ For system To have nontrivial Solution.

$A\,\mathrm{adj}(A) = \mathrm{der}(A) \cdot I$

$\mathrm{adj}(A) \cdot \dfrac{A}{\mathrm{der}(A)} = I$

$A^{-1}\,\mathrm{adj}(A^{-1}) = \mathrm{der}(A^{-1}) \cdot I$

$\mathrm{adj}(A^{-1}) = \quad A\,\mathrm{der}(A^{-1})$

$\left(A\,\mathrm{adj}(A)\right)^{-1} = (\mathrm{der}(A) \cdot I)^{-1}$

$[\mathrm{adj}(A)]^{-1} \cdot A^{-1} = I \cdot \mathrm{der}(A^{-1})$

$[\mathrm{adj}\,A]^{-1} = \mathrm{der}(A^{-1})\,A$

$\mathrm{der}\,A^{T} \cdot \mathrm{der}\,G \cdot \mathrm{der}\,A = G \cdot \mathrm{der}\,G$

$\mathrm{der}(A) \cdot \mathrm{der}\,G \cdot \mathrm{der}\,A = \mathrm{der}\,G$

$[\mathrm{der}(A)]^2 \cdot \mathrm{der}\,G = \mathrm{der}\,G$

$A^T G A = G \; . \; A^T .$

$\exists\, G^{-1}$

a) If A is an invertible matrix, show that $adj(A)$ is invertible with an inverse satisfying $(adj(A))^{-1} = adj(A^{-1})$. **(10 pts)**

b) A matrix A is said to be orthogonal with respect to a matrix G if $A^T G A = G$. Show that if G is invertible, then the determinant of any matrix A orthogonal to G is equal to ± 1. **(10 pts)**

a) $A \, adj(A) = det(A) \cdot I$

$\frac{A}{det(A)} \cdot adj(A) = I$ since A is invirtible matrix $\Rightarrow det(A) \neq 0$

and their exist $[adj(A)]^{-1}$ where:

$[adj(A)]^{-1} = \frac{A}{det(A)}$ $\Rightarrow$ adj(A) is invirtible.

~~since~~ $A^{-1} adj(A^{-1}) = det(A^{-1}) \cdot I$ and A is invertible

Multiply A on both sides: $I \cdot adj(A^{-1}) = A \, det(A^{-1})$ (1)

and $[A \, adj(A)]^{-1} = (det(A) \cdot I)^{-1}$

$\Rightarrow [adj(A)]^{-1} \cdot A^{-1} = I \cdot det(A)^{-1} = det(A^{-1})$

Multiply both sides by A $\Rightarrow$

$[adj(A)]^{-1} = det(A^{-1}) \cdot A$. (2)

$\Rightarrow [adj(A)]^{-1} = adj(A^{-1})$ since

$det(A^{-1}) \cdot A = A \cdot det(A^{-1})$

since $det(A^{-1})$ is a cst. number.

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b) ~~&~~ since A orthogonal to G $\Rightarrow$

$A^T G A = G$ $\Rightarrow$ $det(A^T \cdot G \cdot A) = det(G)$

$det(A^T) \cdot det(G) \cdot det(A) = det(G)$ and $det(A^T) = det(A)$

$\Rightarrow [det(A)]^2 \cdot det(G) = det(G)$ since G is invertible

$\Rightarrow det(G) \neq 0$

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Ex and $G \cdot G^{-1} = I \Rightarrow \det(G \cdot G^{-1}) = \det(I)$

$\det(G) \cdot \det(G^{-1}) = 1$

Multiply Both sides by $\det(G^{-1})$: $\Rightarrow$

$$[\det(A)]^2 \cdot \det(G) \cdot \det(G^{-1}) = \det(G) \cdot \det(G^{-1}) = 1$$

$$[\det(A)]^2 \cdot 1 = 1$$

$$\Rightarrow \det(A) = \pm 1$$

$4[10-6] + 1[9-4]$

$4 \times 4 + 5$

$36 + 5 = 41$

$A^T = -A$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix}$

$4(8-4) + 1(6-4)$

$4(4) + 2$

$16 + 2$

$128 - 4 = 124$

$4(0-12) + 1(18)$

$4[-12+18]$

4×6

$4(-12 + 1(15-4))$

$-12 + 11$

$-6[8-3]$

-30

Evaluate the determinant of the following matrix by a cofactor expansion along a row or column of your choice:

$$\begin{pmatrix} 1 & 3 & 2 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 4 & 2 \\ 2 & 3 & 6 & 5 & 1 \end{pmatrix}.$$

(20 pts)

$6\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

$6[8-3]$

$6 \times 5 = 30$

$$\det(A) = a_{15}\det(A_{15}) - a_{25}\det(A_{25}) + a_{35}\det(A_{35}) - a_{45}\det(A_{45}) + a_{55}\det(A_{55})$$

$$= 0\begin{vmatrix} 4 & 0 & 0 & 1 \\ 3 & 2 & 0 & 2 \\ 2 & 2 & 0 & 4 \\ 2 & 3 & 6 & 5 \end{vmatrix} - 0 + 0 - 2\begin{vmatrix} 1 & 3 & 2 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 2 & 0 & 2 \\ 2 & 3 & 6 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 & 2 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 2 & 0 & 2 \\ 2 & 2 & 0 & 4 \end{vmatrix}$$

$$= -2\left[1\begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 6 & 5 \end{vmatrix} - 3\begin{vmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 2 & 6 & 5 \end{vmatrix} + 2\begin{vmatrix} 4 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 6 \end{vmatrix}\right]$$

$$+1\left[1\begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 0 & 4 \end{vmatrix} - 3\begin{vmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 2 & 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & 0 \end{vmatrix}\right]$$

$$= -2\left[1[\cancel{12} + 1(12)] - 3[-54] + 2(41) - 1[4(12)]\right]$$

$$+ [0 - 0 + 2(18) - 0]$$

$$= \cancel{-2[-90 + 82 - 36] + 36}$$

$$= \cancel{-2[-44] + 36 = 88 + 36 = 124}$$

$$= -2[12 - 90 + 82 - 48] + 36$$

$$= 124$$

$$\cancel{= 247}$$

$$\det(A) = -156$$

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a) Find the inverse matrix of $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{pmatrix}$ by using the **adjoint** matrix. **(10 pts)**

b) Give example of 2×2 matrices A and B such that $\det(A+B) \neq \det(A) + \det(B)$, and another example where $\det(A+B) = \det(A) + \det(B)$. **(10 pts)**

4) $C_{11} = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = -3$; $C_{12} = -\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = -[-2] = 2$

$C_{13} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = 4$; $C_{21} = -\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = -[1 - 2] = 1$

$C_{22} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = -1$; $C_{23} = -\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = -2$

$C_{31} = \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} = -3$; $C_{32} = -\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = 2$

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$C_{33} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = 3 + 2 = 5$.

$C = \begin{pmatrix} -3 & 2 & 4 \\ 1 & -1 & -2 \\ -3 & 2 & 5 \end{pmatrix}$; $\operatorname{adj}(A) = C^T = \begin{pmatrix} -3 & 1 & -3 \\ 2 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$

~~A det(A) =~~ $A \operatorname{adj}(A) = \det(A) I$.

$\Rightarrow A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$ and $\det(A) = 1[-3] + 1[-2] + 1[4]$

$\Rightarrow \det(A) = -3 - 2 + 4 = -5 + 4 = -1$.

$\Rightarrow A^{-1} = \begin{pmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{pmatrix}$

0

b) $\det(A+B) \neq \det(A) + \det(B)$

$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2$ $\det\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix} = 12 - 10 = 2$

$\det\begin{pmatrix} 4 & 0 \\ 2 & 2 \end{pmatrix} =$ ~~28 − 30 = −2~~ 2

$\Rightarrow 2 \neq (2 - 2 = 0)$ $\Leftrightarrow \det(A+B) \neq \det(A) + \det(B)$

An eg. for $\det(A) + \det(B) = \det(A+B)$

is: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $A+B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\det(A) = 1$; $\det(B) = 0$; $\det(A+B) = 1$

$\Rightarrow \det(A+B) = \det(A) + \det(B)$

$1 = 0 + 1 = 1$ ✓