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MAT215: Linear Algebra I

Exam II: 22nd January, 2007
Time: 1 Hour

Name: [redacted].

ID Number: [redacted].

Instructor: [redacted].

1) By using the **determinant** of the coefficient matrix of the following homogeneous system, find a relation between α and β so that the system has nontrivial solutions:

$$
\begin{aligned}
x + y + \alpha z &= 0 \\
x + y + \beta z &= 0 \quad . \qquad \textbf{(20 pts)} \\
\alpha x + \beta y + z &= 0
\end{aligned}
$$

$$= \begin{vmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{vmatrix} =$$

$$= 1\begin{vmatrix} 1 & \beta \\ \beta & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & \beta \\ \alpha & 1 \end{vmatrix} + \alpha\begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix}$$

$$= 1(1-\beta^2) - 1(1-\alpha\beta) + \alpha(\beta - \alpha)$$

$$= 1 - \beta^2 - \cancel{1} + \alpha\beta + \alpha\beta - \alpha^2$$

$$= -\beta^2 + 2\alpha\beta - \alpha^2 = 0 \Rightarrow (\alpha - \beta)^2 = 0$$

$$\Rightarrow \alpha = \beta$$

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2) a) If A is an invertible matrix, show that $adj(A)$ is invertible with an inverse satisfying $(adj(A))^{-1} = adj(A^{-1})$. **(10 pts)**

b) A matrix A is said to be orthogonal with respect to a matrix G if $A^T GA = G$. Show that if G is invertible, then the determinant of any matrix A orthogonal to G is equal to ± 1. **(10 pts)**

a) A is an invertible matrix $adj(A^{-1}) = (adj(A))^{-1}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \Rightarrow C = \begin{pmatrix} c_{11} & & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & & c_{nn} \end{pmatrix}$$

$\det A \cdot Adj = A \cdot I$

$\Rightarrow$

$$adj A = \begin{pmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{pmatrix} \Rightarrow A \cdot I$$

$\det A \; adj A = A^{-1}$

$\Rightarrow adj A = \dfrac{A^{-1}}{\det A}$

b) $A^T GA = G$

G is invertible $\det G \neq 0$

$\det(A^T) = \det(A)$

$\det(A^T G A) = \det G$

$|AGA| = |G|$

$|A^2| = \dfrac{|G|}{|G|} = 1 \Rightarrow |A| = \pm 1$

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O.K.

3) Evaluate the determinant of the following matrix by a cofactor expansion along a row or column of your choice:

$$\begin{pmatrix} 1 & 3 & 2 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 4 & 2 \\ 2 & 3 & 6 & 5 & 1 \end{pmatrix}. \qquad \textbf{(20 pts)}$$

~~We use the 3rd co~~

Cofactor expansion along the 3rd ~~row~~ column : $a_{13}\times c_{13} + a_{23}\times c_{23} + a_{20}\times c_{24} + a_{25}\times c_{25}$

$$\begin{vmatrix} +1 & 3 & 2 & 1 & 0 \\ -4 & 0 & -0 & 1 & 0 \\ +3 & 2 & +0 & 2 & 0 \\ -2 & 2 & -0 & 4 & 2 \\ +2 & -3 & +6 & 5 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 \\ 2 & 2 & 4 & 2 \\ 2 & 3 & 5 & 1 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 & 1 & 0 \\ 4 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 \\ 2 & 2 & 4 & 2 \end{vmatrix}$$

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$$= 2\left(4\begin{vmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 3 & 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 2 & 3 & 1 \end{vmatrix}\right) + 6\left(-4\begin{vmatrix} 3 & 1 & 0 \\ 2 & 2 & 0 \\ 2 & 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix}\right)$$

$$= 2\left(4\times\left(2\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}\right) + 1\left(3\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}\right)\right) + 6\left(-4\left(3\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix}\right) + 1\left(1\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix}\right)\right)$$

$$= 2\left(4\times(2(-6) - 2(-4)) + 1(3(-4) - 2(0))\right) + 6\left(-4(3(4) - 1(4)) + 1\times(1(4) - 3(6))\right) = 2\left((4\times -4) + 1(-12)\right) + 6\left(-4(8) + 1\times(-14)\right)$$

$$= 2(-28) + 6(-56) = -392$$

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4.) a) Find the inverse matrix of $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{pmatrix}$ by using the **adjoint** matrix. **(10 pts)**

b) Give example of 2×2 matrices A and B such that $\det(A+B) \neq \det(A) + \det(B)$, and another example where $\det(A+B) = \det(A) + \det(B)$. **(10 pts)**

a) $C_{11} = \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3$ $\quad C_{21} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -1$ $\quad C_{31} = \begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = -3$

$C_{12} = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2 = 2$ $\quad C_{22} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1$ $\quad C_{32} = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$

$C_{13} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 4$ $\quad C_{23} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -2$ $\quad C_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$

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$C = \begin{pmatrix} 3 & 2 & 4 \\ 1 & -1 & -2 \\ -3 & 2 & 5 \end{pmatrix} \Rightarrow C^T = \mathrm{adj}(A) = \begin{pmatrix} 3 & 1 & -3 \\ 2 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$

$A^{-1} = \frac{1}{\det A} \mathrm{adj}\, A \Rightarrow \det A = 1(-3) - (-1)(-2) + 1(4)$

$= -3 - 2 + 4 = -1$

$\Rightarrow A^{-1} = \frac{1}{-1} \begin{pmatrix} 3 & 1 & -3 \\ 2 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix} = - \begin{pmatrix} 3 & 1 & -3 \\ 2 & -1 & 2 \\ 4 & -2 & 5 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $\det A = 3$ $\quad B = \begin{pmatrix} 5 & 2 \\ 0 & 6 \end{pmatrix} = 30$

$(A+B) = \begin{pmatrix} 6 & 4 \\ 0 & 9 \end{pmatrix} \Rightarrow \det(A+B) = 54$ $\quad \Rightarrow \det(A+B) \neq \det A + \det(B)$

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$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det(A) = 1$ $\quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(B) = -1$

$\det(A) + \det(B) = 1 + (-1) = 0$

$A+B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \det(A+B) = 1 - 1 = 0$

Exercise 5:

skew symm. $A = -A^T$

a) $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = - \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

W is close under $\oplus$: $w_1 + w_2 \in V$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a_1 & c \\ b & d \end{pmatrix} =$

No

W is closed under $\odot$: $k w_1 \in V$

$kw = k \begin{pmatrix} a_1 & b \\ c & d \end{pmatrix}$:

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5) a) Show that the set W of all antisymmetric (or skew-symmetric) 2×2 matrices forms a subspace of the vector space M_{22} of all 2×2 matrices. **(10 pts)**

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b) Consider the set $V = \mathbf{R}^2 = \{(a, b) : a, b \in \mathbf{R}\}$. And consider the two operations $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2 + a_1 a_2,\ b_1 + b_2)$, and $k \cdot (a, b) = (ka, kb)$. Determine whether V, together with these two operations, give a vector space. **(10 pts)**

b) $V = \mathbb{R}^2 = \{a, b ; a, b \in \mathbb{R}\}$ $\quad V_1(a_1, b_1)$, $V_2(a_2, b_2)$, $V_3(a_3, b_3)$

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2 + a_1 a_2, b_1 + b_2)$

$K \cdot (a, b) = ka, kb)$

1 - V is close under ⊕ ; $V_1 + V_2 \in V \Rightarrow (a_1, b_1) + (a_2, b_2) = a_1 + a_2 + a_1 a_2, (b_1 + b_2) \in V$

$V_1 \in V$

$V_2 \in V$ True

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2 - V is close under ⊗ ; $KV_1 \in V \Rightarrow K \cdot (a, b) = (ka, kb) \in V$: True

3 - + is commutative $V_1 + V_2 = V_2 + V_1$

$V_1 + V_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2 + a_1 a_2, b_1 + b_2)$

$V_2 + V_1 = (a_2, b_2) + (a_1, b_1) = (a_2 + a_1 + a_1 a_2, b_2 + b_1) \Rightarrow V_1 + V_2 = V_2 + V_1$ False

4 - + is associative $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$

$(V_1 + V_2) + V_3 = (a_1 + a_2 + a_1 a_2) + (a_3, b_3)$

$V_1 (V_2 + V_3) = (a_1, b_1) + (a_2, b_2) + (a_3, b_3) \Rightarrow$ True

5) $0 + V = V$

$(a', b') + (a_1, b_1) = (a_1, b_1)$

$(a' + a_1 + a' a_1, b' + b_1) = (a_1, b_1)$

$a' + a_1 + a' a = a_1$

$a' + a' a = 0 \Rightarrow a'(1 + a) = 0$, $a' = 0$

$b' + b_1 = b_1$

$\Rightarrow b' = 0 \Rightarrow$ zero term $0(0, 0)$

6) opposite: $-v + v = (0,0)$

$(a'', b'') + (a_1 + b_1) = (0,0)$

$a'' + a_1 + a''a_1, b'' + b_1 = (0,0)$

$a'' + a_1 + a''a_1 = 0$ $\qquad$ $b'' + b_1 = 0$

$b'' = -b_1$

$a''(1 + a_1) + a_1$

$a''(1 + a_1) = -a_1$

$a'' = \dfrac{-a_1}{1 + a_1}$

$-v\left(\dfrac{-a_1}{1+a_1}, -b_1\right)$

false

does not exist if $a_1 = -1$

7) $k(v_1 + v_2) = kv_1 + kv_2$

$k(a_1 + a_2 + a_1a_2, b_1 + b_2) = k(a_1 + a_2 + a_1a_2, k(b_1 + b_2))$

$kv_1 = (ka_1, kb_1) + (ka_2, kb_2)$

$= \cancel{k(a_1 + a_2)}\ ka_1 + ka_2 + ka_1a_2, k(b_1 + b_2)$

$= k(a_1 + a_2 + a_1a_2, k(b_1 + b_2)) =$ True

8- $(k + L) + v = k(L + v)$

$(k + L) + (a_1, b_1) = k(L + (a_1, b_1))$ False

9 $(kL) \cdot v = k \cdot (L \cdot v)$

$= kL)(a_1, b_1) = kLa_1, kLb_1$

$= kLa_1, kLb_1)$ True

10 $1 \cdot v = v$

$1 \cdot (a_1, b_1) = (1a_1, 1b_1)$

$= (a_1, b_1)$ True So it is ~~a vector space~~ it is not a vector s