NDU

Notre Dame University

MAT 215

Linear Algebra I

Exam 1

Wednesday April 14, 2010

Duration: 55 minutes

Name: ______________________

Section: ______________________

Instructor: ______________________

Grade: 86

1) (20 points) Find the general solution of the following system by Gauss-Jordan elimination.

20

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ 3x \qquad\quad - 3w &= -3 \\ -x + 2y - 4z + w &= 1 \end{aligned}$$

$$\begin{pmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ 3 & 0 & 0 & -3 & -3 \\ -1 & 2 & -4 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{r_3 = r_3 - 3r_1 \\ r_4 = r_4 + r_1}]{r_2 = r_2 - 2r_1} \begin{pmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{pmatrix} \xrightarrow{r_4 \leftrightarrow r_2}$$

$$\begin{pmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{r_3 = r_3 - 3r_2 \\ r_4 = r_4 - 3r_2}]{r_1 = r_1 + r_2} \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

~~w~~ so $x - w = -1$ $\quad x = w - 1$

$y - 2z = 0$ $\quad y = 2z$

Let $z = r$ and $w = s$

So $x = s - 1$

$y = 2r$

$z = r$

$w = s$

18

2) **(20 points)** Find the value(s) of k for which the system below has a unique solution, no solution, or infinitely many solutions.

$$x+y+kz=0$$
$$x+2y+3z=k$$
$$x+y+k^2z=2k$$

$$\begin{pmatrix} 1 & 1 & K & 0 \\ 1 & 2 & 3 & K \\ 1 & 1 & K^2 & 2K \end{pmatrix} \xrightarrow[r_3=r_3-r_1]{r_2=r_2-r_1} \begin{pmatrix} 1 & 1 & K & 0 \\ 0 & 1 & 3-K & K \\ 0 & 0 & K^2-K & 2K \end{pmatrix}$$

$$\xrightarrow{r_1=r_1-r_2} \begin{pmatrix} 1 & 0 & 2K-3 & -K \\ 0 & 1 & 3-K & K \\ 0 & 0 & K^2-K & 2K \end{pmatrix}$$

for $K^2-K=0$

$K(K-1)=0$ so $K=0$ or $K=1$ the system has no solution

for $K\neq 0$ and $K\neq 1$ the system has always a unique solution. and there is no value for K that make the system has infinitely many solutions.

3) (18 points) Consider the matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

a) **(9 pts)** Find elementary matrices E_1, E_2, and E_3 such that $E_3E_2E_1A = I$.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 = \frac{r_2}{4}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 = r_2 - \frac{3}{4} r_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_1 = r_1 + 2r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

18

So $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$; $E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) **(9 points)** Use (a) to express A as a product of three elementary matrices (find explicitly these matrices).

$E_3 E_2 E_1 A = I$

E_1 ; E_2 and E_3 are elementary matrices so they are invertible

and $A = E_1^{-1} E_2^{-1} E_3^{-1} I$ waste of time!!

$A = E_1^{-1} E_2^{-1} E_3^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 = 4r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

So $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 = r_2 + \frac{3}{4} r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

So $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 = r_1 - 2r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

So $E_3^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

finally

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4) (21 points) Determine whether each of the following statements is true or not. If you think it is true then give a proof. If you think it is wrong, give a counterexample.

a) A linear system of five equations in two unknowns cannot have a unique solution.

7

False:
$x + y = 1$
$x - y = 2$
$2x - 2y = 4$
$3x + 3y = 3$
$6x - 6y = 12$

This system has a unique solution
$x = \frac{3}{2}$ and $y = \frac{5}{3} - \frac{1}{2}$

b) If A is a square matrix and AA has a column of zeros, then A must have a column of zeros.

7

False: $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ $AA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

3

c) If A and B are $n \times n$ matrices, then $(AB)^2 = A^2B^2$.

$\cancel{(AB)^2 = (AB)(AB) = ABAB}$

False: $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

show why?

5) (21 points) Prove each of the following statements:

a) If A is an $n \times n$ invertible matrix and B is row-equivalent to A, then B is also invertible.

7

We have A invertible and

$E_n \cdots E_2 E_1 B = A$ so $B = E_1^{-1} E_2^{-1} \cdots E_n^{-1} A$

Since $E_1^{-1}, E_2^{-1} \cdots E_n^{-1}$ are invertible so B is invertible because the product of invertible matrices is invertible.

b) If B and C are both inverses of a matrix A, then $B = C$.

6

We have

$BA = I$ and $AC = I$

$(BA)C = C$ why? $B(AC) = B$

and $BAC = BAC$ so $C = B$.

c) Suppose A is a square matrix such that $A^2 = 0$. Show that $(I - A)$ is invertible by finding its inverse.

$A(I-A) = AI - AA = AI - A^2 = A$.