NDU

MAT 215

Linear Algebra I

Exam # 2

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Duration: 75 minutes

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Section: ____________

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Grade: 90

Problem Number	Points	Score
1	14	14
2	16	15
3	14	13
4	12	12
5	20	16
6	8	5
7	16	15
Total	100	

1) (14 points) Evaluate the determinant of the matrix: $A = \begin{bmatrix} 1 & 0 & 4 & 1 & 7 \\ 3 & 5 & 4 & 1 & 7 \\ 2 & 0 & 8 & 2 & 9 \\ 3 & 0 & 8 & 3 & 5 \\ 4 & 2 & 3 & 4 & 1 \end{bmatrix}$.

$$|A| = \begin{vmatrix} 1 & 0 & 4 & 1 & 7 \\ 3 & 5 & 4 & 1 & 7 \\ 2 & 0 & 8 & 2 & 9 \\ 3 & 0 & 8 & 3 & 5 \\ 4 & 2 & 3 & 4 & 1 \end{vmatrix} \quad -R_1 + R_2 \to R_2$$

$$= \begin{vmatrix} 1 & 0 & 4 & 1 & 7 \\ 2 & 5 & 0 & 0 & 0 \\ 2 & 0 & 8 & 2 & 9 \\ 3 & 0 & 8 & 3 & 5 \\ 4 & 2 & 3 & 4 & 1 \end{vmatrix}$$

same column so $\boxed{\det(\,) = 0}$

14

$$= -2 \begin{vmatrix} 0 & 4 & 1 & 7 \\ 0 & 8 & 2 & 9 \\ 0 & 8 & 3 & 5 \\ 2 & 3 & 4 & 1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 4 & 1 & 7 \\ 2 & 8 & 2 & 9 \\ 3 & 8 & 3 & 5 \\ 4 & 3 & 4 & 1 \end{vmatrix}$$

$$= -2 \left(-2 \begin{vmatrix} 4 & 1 & 7 \\ 8 & 2 & 9 \\ 8 & 3 & 5 \end{vmatrix} \right. + 0$$

$$= 4 \begin{vmatrix} 4 & 1 & 7 \\ 8 & 2 & 9 \\ 8 & 3 & 5 \end{vmatrix} \quad -2R_1 + R_2 \to R_2$$

$$= 4 \begin{vmatrix} 4 & 1 & 7 \\ 0 & 0 & -5 \\ 8 & 3 & 5 \end{vmatrix}$$

$$= 4 \left(+5 \begin{vmatrix} 4 & 1 \\ 8 & 3 \end{vmatrix} \right.$$

$$= -20 \begin{vmatrix} 4 & 1 \\ 8 & 3 \end{vmatrix}$$

$$= -20 \left(+12 - 8 \right)$$

$$= -20 \left(4 \right)$$

$$= -80$$

$$A = \begin{bmatrix} 1 & 0 & 4 & 1 & 7 \\ 3 & 5 & 4 & 1 & 7 \\ 2 & 0 & 8 & 2 & 9 \\ 3 & 0 & 8 & 3 & 5 \\ 4 & 2 & \cancel{4}3 & \cancel{3}4 & 1 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 4 & 1 & 7 \\ \cancel{2} & \cancel{5} & \cancel{0} & \cancel{0} & \cancel{0} \\ 2 & 0 & 8 & 2 & 9 \\ 3 & 0 & 8 & 3 & 5 \\ 4 & 2 & 3 & 4 & 1 \end{vmatrix}$$

$$\begin{array}{r} -14 \\ +9 \\ \hline 5 \end{array}$$

$$= -2 \begin{vmatrix} 0 & 4 & 1 & 7 \\ 0 & 8 & 2 & 9 \\ 0 & 8 & 3 & 5 \\ \cancel{2} & \cancel{3} & \cancel{4} & \cancel{1} \end{vmatrix} + 5 \begin{vmatrix} 1 & 4 & 1 & 7 \\ 2 & 8 & 2 & 9 \\ 3 & 8 & 3 & 5 \\ 4 & 3 & 4 & 1 \end{vmatrix}$$

$$= + \cancel{2}4 \begin{vmatrix} 4 & 1 & 7 \\ 8 & 2 & 9 \\ 8 & 3 & 5 \end{vmatrix} \quad \cancel{+81} \; +0 \begin{vmatrix} \cancel{0\,4\,4\,7} \end{vmatrix}$$

$$= 4 \begin{vmatrix} 4 & 1 & \cancel{7} \\ 0 & 0 & 5 \\ 8 & 3 & \cancel{5} \end{vmatrix} \qquad = 4 \left(-5 \begin{vmatrix} 4 & 1 \\ 8 & 3 \end{vmatrix} \right)$$

$$= 4\,(-5\,(12 - 8))$$

$$\begin{array}{r} 16 \\ \times 4 \\ \hline 64 \end{array}$$

$$= 4\,(-5\,(4))$$

$$= 4\,(-20)$$

$$= -80.$$

2) **(16 points)** Let $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Given that $\det(A)=5$, find each of the following:

a) **(4 points)** $\det(4A^{-1})$

$$\det(4A^{-1}) = 4^3 \det(A^{-1})$$
$$= 4^3 \frac{1}{\det(A)}$$
$$= \frac{4^3}{5} = 64/5.$$

4 ✓

b) **(6 points)** $\begin{vmatrix} d & e & f \\ 4a & 4b & 4c \\ g+2d & h+2e & i+2f \end{vmatrix}$

5

$$= 4\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$
$$= 4 \times \det(A)$$
$$= 20$$

c) **(6 points)** $\begin{vmatrix} a+2d & 2a+d & g \\ b+2e & 2b+e & h \\ c+2f & 2c+f & i \end{vmatrix}$

we have $\det(A^T) = \det(A)$

6

$$= \begin{vmatrix} a & 2a+d & g \\ b & 2b+e & h \\ c & 2c+f & i \end{vmatrix} + \begin{vmatrix} 2d & 2a+d & g \\ 2e & 2b+e & h \\ 2f & 2c+f & i \end{vmatrix}$$
$$= \begin{vmatrix} a & 2a & g \\ b & 2b & h \\ c & 2c & i \end{vmatrix} + \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} + \begin{vmatrix} 2d & 2a & g \\ 2e & 2b & h \\ 2f & 2c & i \end{vmatrix} + \begin{vmatrix} 2d & d & g \\ 2e & e & h \\ 2f & f & i \end{vmatrix}$$
$$= 0 + \det(A^T) + 4\begin{vmatrix} d & a & g \\ e & b & h \\ f & c & i \end{vmatrix} + 0$$
$$= 5 - 4\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$
$$= 5 - 4\det(A^T)$$
$$= 5 - 20$$
$$= -15.$$

a b c
d e f
g h i

d a g
e b h
f c i

a d g
b e h
c f i

3) **(14 points)** For the matrix $A=\begin{bmatrix}1 & 2 & 3\\ -2 & -1 & 0\\ 5 & 4 & 2\end{bmatrix}$, find:

a) **(10 points)** $adj(A)$

$M_{11} = -2$ $\quad C_{11} = -2$
$M_{12} = -4$ $\quad C_{12} = 4$
$M_{13} = -3$ $\quad C_{13} = -3$
$M_{21} = -8$ $\quad C_{21} = 8$
$M_{22} = -6$ $\quad C_{22} = -6$
$M_{23} = -6$ $\quad C_{23} = 6$
$M_{31} = 3$ $\quad C_{31} = 3$
$M_{32} = 6$ $\quad C_{32} = -6$
$M_{33} = 3$ $\quad C_{33} = 3$

9

$$adj(A) = \begin{pmatrix} -2 & 8 & 3 \\ 4 & -6 & -6 \\ -3 & 6 & 3 \end{pmatrix}$$

b) **(4 points)** Use $adj(A)$ to find A^{-1}.

4

$$A^{-1} = \frac{1}{det(A)} \, adj(A)$$

$$det(A) = 1\begin{vmatrix}-1 & 0\\ 4 & 2\end{vmatrix} - 2\begin{vmatrix}-2 & 0\\ 5 & 2\end{vmatrix} + 3\begin{vmatrix}-2 & -1\\ 5 & 4\end{vmatrix}$$
$$= -2 \quad -2(-4) \quad +3(-8+5)$$
$$= -2 \quad +8 \quad -9$$
$$= -3$$

$$A^{-1} = -1/3 \begin{pmatrix} -2 & 8 & 3 \\ 4 & -6 & -6 \\ -3 & 6 & 3 \end{pmatrix}$$

=>

$$\therefore \quad A^{-1} = \begin{pmatrix} 2/3 & -8/3 & -1 \\ -4/3 & 2 & 2 \\ 1 & -2 & -1 \end{pmatrix}$$

4) (12 points)

a) **(6 points)** Use determinants to find the values of k for which the following homogeneous system will have non-trivial solutions:

$$(k-5)x-2y=0$$
$$x+(k-2)y=0$$

$$\begin{vmatrix} k-5 & -2 \\ 1 & k-2 \end{vmatrix} = 0$$ // other than the trivial solution $\boxed{\det(A)=0}$

6

$(k-5)(k-2)+2=0$

$k^2-2k-5k+10+2=0$

$k^2-7k+12=0$

$D=b^2-4ac$
$=49-4(1)(12)$
$=49-42$
$=1.$

$\boxed{k=4}$ $\boxed{k=3}$ for $k=4$, $k=3$.

b) **(6 points)** Use Cramer's rule to solve the following system for z only:

$$x+2y+3z=1$$
$$y+2z=2$$
$$3x \quad -z=3$$

6

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{pmatrix}$$

$$\det(A) = 1\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}$$
$$= -1 - 2(-6) + 3(-3)$$
$$= -1 + 12 - 9$$
$$= 2$$

$$z = \frac{|A_3|}{|A|}$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{vmatrix}$$
$$= 1\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}$$
$$= 3 - 2(-6) - 3$$
$$= 12$$

$$\boxed{z = \frac{12}{2} = 6}$$

$\Delta = b^2 - 4ac$
$= 49 - 48$

$x = \frac{-b + \sqrt{\Delta}}{2a}$

$x = \frac{7}{2}$

4×12
48

$x = \frac{-b \pm \sqrt{\Delta}}{2a}$

$= \frac{7+1}{2}$

$= 4$

$\frac{7-1}{2}$

5) **(20 points)** Given that A, B and C are $n \times n$ matrices. Prove each of the following statements:

a) **(4 points)** Suppose B is obtained from A by multiplying the i^{th} row of A by k. Use the definition of determinants to show that $\det(B) = k\det(A)$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}. \end{pmatrix}$$

1

$\boxed{E_1 A = B}$.

$\boxed{\det(E_1) = k}$

$\det(E_1 A) = \det(B)$.

$\det(B) = \det(E_1)\det(A)$

$\boxed{\det(B) = k\det(A)}$

b) **(4 points)** If $\det(A) \neq 0$ and $AB = AC$, then $B = C$.

If $\det(A) \neq 0$

then A is invertible.

4

$A^{-1}A \quad AA^{-1} B = AA^{-1} C$.

$IB = IC$

$\boxed{B = C}$.

c) **(6 points)** $\det(AB) = \det(A) \cdot \det(B)$.

A is invertible so A can be written as a product of elementary matrix

$A = E_1 E_2 \text{ - - - - - - } E_n$.

5

$AB = E_1 E_2 \text{ - - - - - - - - } E_2 B$

$\det(AB) = \det(E_1 E_2 \text{ - - - - } E_2 B) =$

$\det(AB) = \det(E_1 E_2 \text{ - - - } E_2)\det(B)$

$\boxed{\det(AB) = \det(A)\det(B)}$

d) **(6 points)** Let A be an $n \times n$ matrix $\boxed{\text{with } n \text{ odd}}$. Prove that if A is skew-symmetric, then $\det(A) = 0$. (Recall: A is skew-symmetric if $A^T = -A$).

$\boxed{A^T = -A}$

we know that $\det(A^T) = \det(A)$.

6

$\det(A^T) = \det(-A)$.

$= (-1)^n \det(A)$. and n is odd. so

$\det(A^T) = -\det(A)$.

$\det(A^T) + \det(A) = 0$

$2\det(A) = 0$

$\Rightarrow \boxed{\det(A) = 0}$

6) **(8 points)** Let A be an $n \times n$ matrix which is not invertible.

a) **(4 points)** Show that $A\,adj(A) = O_{n \times n}$.

4 ✓

$$A\,adj(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ C_{13} & \vdots & & \vdots \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

If $i = j$ then you are multiplying
The entries of the row with its cofactor
if $i \neq j$ then you are multiplying
the entries of the row with the
cofactor of another row
so the sum of the product $= 0$

$\Rightarrow$ we have then

$$= \begin{pmatrix} \det(A) & 0 & 0 & 0 & \cdots \\ 0 & \det(A) & 0 & 0 & \cdots \\ 0 & 0 & \ddots & & \\ 0 & 0 & \cdots & & \det(A) \end{pmatrix}$$

$A\,adj(A) = \det(A)\, I \Rightarrow \det(A) = 0$

$\Rightarrow \boxed{A\,adj(A) = O_{n \times n}}$

b) **(4 points)** Use part (a) to deduce that $adj(A)$ is not invertible.

???

if $A\,adj(A) = 0$

~~then~~ ~~$\det(A\,adj(A))$~~ $= 0$

~~$\det(A)\det(adj\,A)$~~ $= 0$

~~det~~

1

if A has a row of zeros
then $adj(A)$ will have
a ~~row~~ column of 0
So $adj(A)$
is singular

$A\,adj(A) = 0$

$\det(A)\,adj(A) = 0$

$\det(A)\det(adj(A)) = 0$

then $\det(adj(A)) = 0$

$adj(A)$ is not invertible.

7) **(16 points)** Let $V = \mathbb{R}^2$, the set of all pairs of real numbers, with the operations:

$$(x,y)+(x',y')=(xx',2yy')$$
$$k(x,y)=(kx,ky)$$

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Check **all 10 axioms** that must hold for V to be a vector space, and determine which axioms are true and which are false. Then determine whether or not V is a vector space.

$V = \{ (x,y)+(x',y') = (xx', 2yy')$, $k(x,y) = (kx,ky) \}$.

axiom 1: $\vec{u}(x,y) \in V$. $\vec{v}(x',y') \in V$. $\vec{u}+\vec{v} = (x,y)+(x',y') = (xx', 2yy') \in V$.
then axiom 1 is satisfied ✓

axiom 2: $\vec{u}+\vec{v} = (x,y)+(x',y') = (xx', 2yy')$ $\vec{v}+\vec{u} = \vec{u}+\vec{v}$
$\vec{v}+\vec{u} = (x',y')+(x,y) = (xx', 2yy')$ axiom 2 is satisfied ✓

axiom 3: $\vec{w}(x'',y'')$
$(\vec{u}+\vec{v})+\vec{w} = (xx', 2yy') + (x'',y'')$
$= (xx'x'', 4yy'y'')$ ✓
$\vec{u}+(\vec{v}+\vec{w}) =$ ~~$(x$~~ $(x,y) + (x'x'', 2y'y'')$
$= (xx'x'', 4yy'y'') \Rightarrow (\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.

axiom 4: $\vec{0}(a,b)$ $\vec{v}(x,y)$
$\vec{0}+\vec{v} = (ax, 2by)$ ✓
$\vec{v}+\vec{0} = (ax, 2by)$.
$\vec{0}+\vec{v} = \vec{v}+\vec{0} = (ax, 2by)$ $\boxed{\vec{0}(1, 1/2) \in V}$
$ax = x$.
$\boxed{a=1}$
$2by = y$. axiom 4 is satisfied
$2b = 1$
$\boxed{b = 1/2}$

axiom 5: $(-\vec{v})(e,f)$
$(e,f)+(x,y)$
$(-\vec{v})+(\vec{v}) = (\vec{v})+(-\vec{v}) = (ex, 2fy)$
$\boxed{(-\vec{v})\ (1/x, 1/4y) \in V}$
$x = y = 0$?
$ex = 1$
$\boxed{e = 1/x}$ axiom 5 is satisfied ✓
$2fy = 1/2$.
$2f = \frac{1}{2y}$.
$\boxed{f = 1/4y}$

axiom 6: $k\vec{v} = k(x,y) = (kx,ky) \in V$
axiom 6 satisfied ✓

axiom 7: $k(\vec{u}+\vec{v}) \overset{?}{=} k\vec{u}+k\vec{v}$.
$k(\vec{u}+\vec{v}) = k((x,y)+(x',y')) = k(xx', 2yy')$ ✓
$= (kxx', 2kyy')$

$k\vec{u}+k\vec{v} = (kx,ky)+(kx',ky')$
$= (k^2xx', 2k^2yy') \neq (kxx', 2kyy')$ axiom 7 is not satisfied.

→

axiom 8. $(k+l)\vec{u} \overset{?}{=} k\vec{u} + l\vec{u}$

$$(k+l)\vec{u} = ((k+l)x, (k+l)y)$$
$$= (kx + lx, ky + ly)$$

✓ $k\vec{u} + l\vec{u} = (kx, ky) + (lx, ly)$

$$= (kl x^2, 2kly^2) \neq (k+l)\vec{u}$$

axiom 8 is not satisfied.

axiom 9 $(k)l\vec{u} \overset{?}{=} (kl)\vec{u}$.

$(k)(lx, ly) = (klx, kly)$ ✓

$(kl)\vec{u} = (klx, kly)$ so it is satisfied

axiom 10 $1.\vec{u} \overset{?}{=} \vec{u}$

$1.(x,y) = (1x, 1y) = (x,y)$ ✓

so axiom 10 is satisfied

But this is not a vector space because axiom 7 and 8 are not satisfied.