Exam #2

Grade 85

1) Consider the matrix $A=\begin{pmatrix}1 & -2 & 3\\ 6 & 7 & -1\\ -3 & 1 & 4\end{pmatrix}$. (20 points)

a) Find $\det(A)$ by using a cofactor expansion along any row or column.

b) Find $\mathrm{adj}(A)$

c) Find A^{-1} using parts *a* and *b*.

a) $\det(A) = c_{11}a_{11} + c_{12}a_{12} + c_{13}a_{13}$

$\det(A) = (-1)^{1+1}\times 1 \times \det\begin{pmatrix}7 & -1\\ 1 & 4\end{pmatrix} + (-1)^{1+2}\times(-2)\times\det\begin{pmatrix}6 & -1\\ -3 & 4\end{pmatrix} + (-1)^{1+3}\times 3\times\det\begin{pmatrix}6 & 7\\ -3 & 1\end{pmatrix}$

$\det(A) = 29 + 2\times(6\times 4 - (-3)\times(-1)) + 3\times(6\times 1 - (-3)\times 7)$

$\det(A) = 29 + 42 + 81 = 152$.

b) $C^t = \mathrm{adj}(A)$.

$C_{11} = (-1)^{1+1}\times\begin{pmatrix}7 & -1\\ 1 & 4\end{pmatrix} = 29$. $C_{12} = (-1)^{1+2}\begin{pmatrix}6 & -1\\ -3 & 4\end{pmatrix} = -21$.

$C_{13} = (-1)^{1+3}\begin{pmatrix}6 & 7\\ -3 & 1\end{pmatrix} = 27$. $C_{21} = (-1)^{2+1}\begin{pmatrix}-2 & 3\\ 1 & 4\end{pmatrix} = +11$.

$C_{22} = (-1)^{2+2}\begin{pmatrix}1 & 3\\ -3 & 4\end{pmatrix} = 13$. $C_{23} = (-1)^{2+3}\begin{pmatrix}1 & -2\\ -3 & 1\end{pmatrix} = +5$.

$C_{31} = (-1)^{3+1}\begin{pmatrix}-2 & 3\\ 7 & -1\end{pmatrix} = -19$. $C_{32} = (-1)^{3+2}\begin{pmatrix}1 & 3\\ 6 & -1\end{pmatrix} = +19$.

$C_{33} = (-1)^{3+3}\begin{pmatrix}1 & -2\\ 6 & 7\end{pmatrix} = 19$.

$\mathrm{adj}(A) = C^t = \begin{pmatrix}29 & -21 & 27\\ 11 & 13 & 5\\ -19 & 19 & 19\end{pmatrix}^t = \begin{pmatrix}29 & 11 & -19\\ -21 & 13 & 19\\ 27 & 5 & 19\end{pmatrix}$

20

c) $A^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)} = \dfrac{1}{152}\begin{pmatrix}29 & 11 & -19\\ -21 & 13 & 19\\ 27 & 5 & 19\end{pmatrix}$.

$$A^{-1} \operatorname{adj}(A^{-1}) = \det(A^{-1}) = \frac{1}{\det(A)}$$

$$x(A) \qquad \operatorname{adj}(A^{-1}) = \frac{A}{\det(A)}$$

$$\operatorname{adj}(A^{-1})^{-1} \operatorname{adj}(\operatorname{adj} A^{-1})^{-1} = \det \operatorname{Adj}(A^{-1})$$

$$(\operatorname{adj}(A))^{-1} A^{-1} = (\det(A))^{-1} = \frac{1}{\det(A)}$$

2) Let A be an invertible matrix.

a) Show that $[\mathrm{adj}(A)]^{-1} = \dfrac{A}{\det(A)}$. (8 points)

b) Show that $\mathrm{adj}(A^T) = [\mathrm{adj}(A)]^T$. (8 points)

a) $\exists A^{-1} \Leftrightarrow \det(A) \neq 0$.

~~$A \cdot \mathrm{adj}(A) = \det(A) \cdot I$~~ $\qquad \det(A^{-1}) = \dfrac{1}{\det(A)}$

~~$A^{-1} \mathrm{adj}(A^{-1}) = \det(A^{-1})$~~ ~~multiplying both sides by A and replacing $\det(A^{-1})$, we get~~

~~$A \cdot \mathrm{adj}(A^{-1}) = \dfrac{1}{\det(A)} \cdot A$~~

~~$\mathrm{adj}(A^{-1}) = \dfrac{A}{\det(A)} \Rightarrow [\mathrm{adj}(A)]^{-1} = \dfrac{A}{\det(A)}$~~

1

$A \,\mathrm{adj}(A) = \det(A) \cdot I$.

$(A \,\mathrm{adj}(A))^{-1} = (\det(A) \cancel{^{-1}} I)^{-1}$

$(\mathrm{adj}(A))^{-1} \cdot A^{-1} = \dfrac{1}{\det(A)} \cdot I$ multiplying by A on both sides: $\boxed{[\mathrm{adj}(A)]^{-1} = \dfrac{A}{\det(A)}}$.

b) ~~$\mathrm{adj}\, A$~~ $\mathrm{adj}(A^t) = [\mathrm{adj}(A)]^t$?

$A^t \mathrm{adj}(A^t) = \det(A^t) = \det(A)$ (1) since $\det(A^t) = \det(A)$.

and $A \,\mathrm{adj}(A) = \det(A)$.

$(A \,\mathrm{adj}(A))^t = (\det(A))^t$

8

$A^t \mathrm{adj}(A)^t = \cancel{(\det(A^t))} \det(A^t) = \det(A)$. (2)

(1) = (2) = $\det(A)$.

$A^t \mathrm{adj}(A^t) = A^t \mathrm{adj}(A)^t$. since A is invertible then $\exists A^{-1}$, then $\exists (A^{-1})^t$. multiplying by $(A^t)^{-1}$ on both sides, we get:

$\boxed{\mathrm{adj}(A^t) = \mathrm{adj}(A)^t.}$

3) Use Cramer's rule to solve only for y (without solving for x or z) in the system

$$\begin{aligned} x-4y+z&=6\\ 4x-y+2z&=-1\\ 2x+2y-3z&=-20 \end{aligned}$$

(20 points)

$$A=\begin{pmatrix} 1 & -4 & 1\\ 4 & -1 & 2\\ 2 & 2 & -3 \end{pmatrix}$$

$$y=\frac{\det(A_2)}{\det(A)}=\frac{\det\begin{pmatrix} 1 & 6 & 1\\ 4 & -1 & 2\\ 2 & -20 & 3 \end{pmatrix}}{\det(A)}$$

$$* \det(A)=1\times\big((-1)(-3)-2\times 2\big)-(-4)\times\big(4\times(-3)-2\times 2\big)+1\times\big(4\times 2-2\times(-1)\big)$$

$$\det(A)=(3-4)-64+10=-1-64+10=-55.$$

$$* \det\begin{pmatrix} 1 & 6 & 1\\ 4 & -1 & 2\\ 2 & -20 & -3 \end{pmatrix}=1\times(-3+20\times 2)-6\times(4\times 3-2\times 2)+1\times\big(4\times(-20)-2\times(-1)\big)$$

18

$$\det\begin{pmatrix} 1 & 6 & 1\\ 4 & -1 & 2\\ 2 & -20 & 3 \end{pmatrix}=-3+40-48-78=\cancel{-89}\ 61$$

$$y=\frac{-89}{-55}=\boxed{\frac{\cancel{89}\ 61}{-55}}$$

4) a) Consider the vector space P_4. Determine whether the subset $W = \{ax^4 + cx^2 + e\}$ of P_4 consisting of all polynomials of *even* power, is a subspace. (8 points)

b) Consider the vector space $\mathbb{R}^3$. Determine whether the subset W of $\mathbb{R}^3$ consisting of all vectors (x, y, z) such that $x > 3$, is a subspace. (8 points)

c) Determine whether the vector $(1, 3, 1)$ in $\mathbb{R}^3$ belongs to the subspace spanned by $v_1 = (1, -1, 1)$ and $v_2 = (1, 0, 1)$. (8 points)

a) $W = (ax^4 + cx^2 + e)$.

let $W \in P_4$.

$W_1 = \{a_1 x^4 + c_1 x^2 + e_1\}$ $W_2 = \{a_2 x^4 + c_2 x^2 + e_2\}$.

X

$* \; W_1 + W_2 = x^4 \underbrace{(a_1 + a_2)}_{a} + x^2 \underbrace{(c_1 + c_2)}_{c} + \underbrace{(e_1 + e_2)}_{e}$.

then $W_1 + W_2 = a x^4 + c x^2 + e$.

W is closed under "+" in P_4.

$* \; K W_1 = K a_1 x^4 + K c_1 x^2 + K e_1 = K(\underbrace{a_1}_{a} x^4 + \underbrace{c_1}_{c} x^2 + \underbrace{e_1}_{e})$.

then, W is closed under "." in P_4. and W is a subspace in P_4.

b) let $W \in \mathbb{R}^3$. $W = \{(x, y, z) / x > 3\}$.

$* \; W_1 = \{(x_1, y_1, z_1)\}$ $W_2 = \{(x_2, y_2, z_2)\}$.

8

$* \; W_1 + W_2 = \{(x_1 + x_2), (y_1 + y_2), (z_1 + z_2)\}$.

since $x_1 > 3$ and $x_2 > 3$ then, $x_1 + x_2 > 3$. then $W_1 + W_2 = (x, y, z)$

W is closed under "+" in $\mathbb{R}^3$.

$* \; K W_1 = (K x_1, K y_1, K z_1)$.

$x_1 > 3$ but $K x_1$ can be negatif if $K < 0$ then W is not closed under "." in $\mathbb{R}^3$ and W is not subset of $\mathbb{R}^3$.

c) →

→ c) does $v_3(1,3,1)$ in R^3 belong to subspace spanned by $v_1=(1,-1,1)$ $v_2=(1,0,1)$?

$$\cancel{k_1 v_1 + k v_2 = (k,-k,k)+(k,0,k) = (2k,-k,2k) = v}$$

$$a_1 v_1 + a_2 v_2 = (a_1,-a_1,a_1) + (a_2,0,a_2) = (a_1+a_2, -a_1, a_1+a_2) = v(x,y,z)$$

if $v_3(1,3,1)$ is coherent with $v(a_1+a_2, -a_1, a_1+a_2)$ then v_3 belongs to this subspace.

9

$$\begin{cases} a_1+a_2 = 1 \Rightarrow a_2 = 1+3 = 4 \\ -a_1 = 3 \Rightarrow a_1 = -3 \\ a_1+a_2 = 1 \end{cases}$$

$3v_1 + 4v_2 = v_3$ then v_3 belongs to this subspace.

5) a) Determine whether the set $\mathbf{R}^3$ with the addition rule defined by $(x_1, y_1, z_1)+(x_2, y_2, z_2)=(x_1+y_2,\ x_2+y_1,\ z_1+z_2)$ and the usual multiplication by numbers, is a vector space or not. (10 points)

b) Use determinants to find the relation between α and β so that the following homogeneous system has a nontrivial solution: (10 points)

$$x+y+\alpha z=0$$
$$x+y+\beta z=0$$
$$\alpha x+\beta y+z=0$$

a) let $v_1(x_1, y_1, z_1)$.
let $v_2(x_2, y_2, z_2)$

for the set V to be a vector space, 3 conditions sould be satisfied:

* $v_1+v_2=(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+y_2,\ x_2+y_1,\ z_1+z_2)$
V is closed under "+" in R^3.

3

* $Kv_1=(Kx_1, Ky_1, Kz_1)$ for any K, V is closed under "." in R^3.

* $v_0(0,0,0)\in V$.
$v_0 \cdot v_1=(0,0,0)\cdot(x_1,y_1,z_1)=0$

$0 \cdot v_1=0$.

then this set V is a vector space. No

b) $A=\begin{pmatrix}1 & 1 & \alpha\\ 1 & 1 & \beta\\ \alpha & \beta & 1\end{pmatrix}$ to have a non-trivial solution, this matrix (corresponding to the system) should have $\det(A)\neq 0$ so that A^{-1} would exist and rref(A) would exist:

6

$\det(A)=1(1-\beta^2)-1(1-\alpha\beta)+\alpha(\beta-\alpha)\neq 0$.

$\det(A)=1-\beta^2-1+\alpha\beta+\alpha\beta-\alpha^2\neq 0$

$\boxed{\beta^2+\alpha^2-2\alpha\beta \neq 0}$ $=$