NDU

MAT 215

Linear Algebra I

Exam # 2

Wednesday January 12, 2005

Duration: 75 minutes

Name: Kovas Stephanie

Section: C (MWF 10:00-11:00)

Instructor: Dr Saade

Grade: ____________

Problem Number	Points	Score
1	20	20
2	20	20
3	20	17
4	25	24
5	15	12
Total	100	93

Excellent

1) (20 points)

a) **(14 points)** Evaluate the determinant of the matrix: $A = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 1 & 2 & 6 & 3 \\ 0 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 8 & 2 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 1 & 2 & 6 & 3 \\ 0 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 1 & 2 & 6 & 3 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & -4 & -6 & 4 & -10 \\ 0 & 0 & 1 & 8 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 2 & 6 & 3 \\ 2 & 3 & 4 & 5 \\ -4 & -6 & 4 & -10 \\ 0 & 1 & 8 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 2 & 6 & 3 \\ 0 & -1 & -8 & -1 \\ 0 & 2 & 28 & 2 \\ 0 & 1 & 8 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -8 & -1 \\ 2 & 28 & 2 \\ 1 & 8 & 2 \end{vmatrix}$$

$= (-1)(28)(2) + (-8)(2)(1) + (-1)(2)(8) - (-8)(2)(2) - (-1)(2)(8) - (-1)(28)($

$= -56 - 16 - 16 + 32 + 16 + 28$

$= -12$

14

b) **(6 points)** Use determinants to find the values of t for which the matrix $A = \begin{bmatrix} t & 2 \\ 1 & (t-1) \end{bmatrix}$ is invertible.

A is invertible det (A) ≠ 0.

$\begin{vmatrix} t & 2 \\ 1 & (t-1) \end{vmatrix} \neq 0.$

$t(t-1) - 2 \neq 0$

$t^2 - t - 2 \neq 0.$

$(t - \frac{1}{2})^2 - \frac{9}{4} \neq 0$

$(t - \frac{1}{2} - \frac{3}{2})(t - \frac{1}{2} + \frac{3}{2}) \neq 0$

$(t-2)(t+1) \neq 0$

$t \neq 2 \quad t \neq -1$

For $t \neq 2$ and $t \neq -1$ the matrix A is invertible

6

2) **(20 points)**

a) **(6 points)** Use Cramer's rule to solve the following system **for z only**:

$$\begin{aligned} x + 2y + 2z &= 3 \\ 2x + 5y + 3z &= 3 \\ x \qquad + 8z &= 17 \end{aligned}$$

$A|\vec{b} = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 2 & 5 & 3 & 3 \\ 1 & 0 & 8 & 17 \end{array}\right)$

$\det(A) = (1)(5)(8) + (2)(3)(1) + (2)(2)(0) - (2)(2)(8) - (1)(3)(0) - (2)(5)(1)$

$= 40 + 6 + 0 - 32 - 10$

$= 4$

$z = \dfrac{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 17 \end{vmatrix}}{\det(A)}$

Or $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 17 \end{vmatrix} = (1)(5)(17) + (2)(3)(1) + (3)(2)(0) - (2)(2)(17) - (1)(3)(0) - (3)(5)(1)$

$= 85 + 6 - 68 - 15$

$= 8$

6

$z = \frac{8}{4} = 2$

b) **(14 points)** For the matrix $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$ find:

i) $adj(A)$

$adj(A) = C^T$

$C_{11} = + \begin{vmatrix}5 & 3\\ 0 & 8\end{vmatrix} = 40$ $\quad C_{21} = - \begin{vmatrix}2 & 2\\ 0 & 8\end{vmatrix} = -16$ $\quad C_{31} = + \begin{vmatrix}2 & 2\\ 5 & 3\end{vmatrix} = -4$

$C_{12} = - \begin{vmatrix}2 & 3\\ 1 & 8\end{vmatrix} = -13$ $\quad C_{22} = + \begin{vmatrix}1 & 2\\ 1 & 8\end{vmatrix} = 6$ $\quad C_{32} = - \begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} = +1$

$C_{13} = + \begin{vmatrix}2 & 5\\ 1 & 0\end{vmatrix} = -5$ $\quad C_{23} = - \begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix} = 2$ $\quad C_{33} = + \begin{vmatrix}1 & 2\\ 2 & 5\end{vmatrix} = 1$

$$C = \begin{bmatrix}40 & -13 & -5\\ -16 & 6 & 2\\ -4 & 1 & 1\end{bmatrix} \qquad adj(A) = C^T = \begin{bmatrix}40 & -16 & -4\\ -13 & 6 & 1\\ -5 & 2 & 1\end{bmatrix}$$

ii) Use $adj(A)$ to find A^{-1}.

$$A^{-1} = \frac{1}{\det A}\, adj(A)$$

$\det(A) = 4$ (see 2) a))

$$A^{-1} = \frac{1}{4}\begin{bmatrix}40 & -16 & -4\\ -13 & 6 & 1\\ -5 & 2 & 1\end{bmatrix} = \begin{bmatrix}10 & -4 & -1\\ -13/4 & 3/2 & 1/4\\ -5/4 & 1/2 & 1/4\end{bmatrix}$$

14

3) **(20 points)** Let A be an $n\times n$ matrix. Prove each of the following:

a) **(4 points)** If A has two identical rows then $\det(A)=0$.

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \det(A)= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ \times(-1) \\ +L_k \\ \\ \end{matrix} \qquad \det(A)= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$\det(A)=0$ since we will have one row of zeroes and to calculate the determinant we must use this row.

4

b) **(6 points)** A is invertible if and only if $\det(A)\neq 0$.

A is invertible $\Leftrightarrow \det(A)\neq 0$

① If A is invertible

$$AA^{-1}=I$$
$$\det(AA^{-1})=\det(I)$$
$$\det(A)\det(A^{-1})=1.$$
$$\Rightarrow \det(A)\neq 0.$$

You can't do this!!

② If A is not invertible. The reduced form (R) of A will contain one row of zeros. $\det(R)=\det(E_p)\det(E_{p-1})\dots\det(A)$.

$$0=\det(E_p)\det(E_{p-1})\dots\det(A)$$

since $\det(E_i)\neq 0$ for any i. Then $\det(A)=0$.

3

c) **(4 points)** IF $A^T=A^{-1}$ then $\det(A)=\pm 1$.

$$A^T=A^{-1}$$
$$\det(A^T)=\det(A^{-1})$$

but $\det(A^T)=\det(A)$

and $\det(A^{-1})=\dfrac{1}{\det(A)}$

Then $\det(A)=\dfrac{1}{\det(A)}$

$[\det(A)]^2=1. \quad \therefore \det(A)=\pm 1$

4

d) **(6 points)** $\det(adj(A))=[\det(A)]^{n-1}$.

$$A^{-1}=\frac{1}{\det(A)}\cdot adj(A)$$

$$\det(A^{-1})=\det\left[\frac{1}{\det(A)}\cdot adj(A)\right]$$

$$\frac{1}{\det(A)}=\left(\frac{1}{\det(A)}\right)^n\det(adj(A))$$

$$\det(adj(A))=\frac{(\det(A))^n}{\det(A)}$$

6

4) (25 points)

a) **(14 points)** Consider R^3 under the operations of regular vector addition and scalar multiplication. Determine whether each of the following is a subspace of R^3:

i) $U=\{(x,2x,3x)\in R^3 \mid x\in R\}$

Let $\vec{u} = (x_1, 2x_1, 3x_1) \in U$
$\vec{v} = (x_2, 2x_2, 3x_2) \in U$

$\vec{u}+\vec{v} = (x_1+x_2,\ 2x_1+2x_2,\ 3x_1+3x_2)$
$\vec{u}+\vec{v} = (x_1+x_2,\ 2(x_1+x_2),\ 3(x_1+x_2))$
$\Rightarrow \vec{u}+\vec{v} \in U$

Let k be a scalar
$k\vec{u} = (kx_1,\ k(2x_1),\ k(3x_1))$
$k\vec{u} = (kx_1,\ 2(kx_1),\ 3(kx_1)) \in U$ $\quad\therefore$ U is a subspace of R^3

ii) $V=\{(x,y,z)\in R^3 \mid x+y+z=3\}$

Let $\vec{u} = (x_1, y_1, z_1) \in V$
$\vec{v} = (x_2, y_2, z_2) \in V$

$\vec{u}+\vec{v} = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$
$(x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1)+(x_2+y_2+z_2) = 3+3 = 6 \neq 3.$
$\Rightarrow \vec{u}+\vec{v} \notin V$
$\therefore$ V is not a subspace of R^3

iii) $W=\{(x,y,z)\in R^3 \mid x=t,\ y=t^2,\ z=0\}$

Let $\vec{u} = (x_1, y_1, z_1) \in V$ $\quad x_1=t \quad y_1=t^2 \quad z=0$
$\vec{v} = (x_2, y_2, z_2) \in V.$ $\quad x_2=t \quad y_2=t^2 \quad z=0.$

$\vec{u}+\vec{v} = (x_1+x_2,\ y_1+y_2,\ z_1+z_2).$

$x_1+x_2 = 2t$, $\ y_1+y_2 = 2t^2 \neq (2t)^2 = 4t^2.$
$\vec{u}+\vec{v} \notin W$
$\therefore$ W is not a subspace

14

b) **(5 points)** Consider the vector space $V = P_4$ of polynomials of degree less than or equal to 4, under regular polynomial addition and scalar multiplication. Let W be the subset of V consisting of polynomials of degree exactly 4.
That is: $p(x) \in W \Leftrightarrow p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_4 \neq 0$. Determine whether W is a subspace of P_4.

$q(x) = 2x^4 + 3x^3 - 2x^2 + x - 5 \in W$
$p(x) = -2x^4 + 6x^3 + 2x^2 + 10 \in W$
$p(x) + q(x) = 9x^3 + x + 5 \notin W$

$\therefore$ W is not a subspace of P_4

5

c) **(6 points)** Find the null space of the matrix $A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 5 & 8 \\ 3 & 9 & 6 & 6 \end{bmatrix}$.

The null space of A is the solution of the augmented matrix $A \mid \vec{0}$. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$\times(-3)$ $\times(-2)$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 2 & 6 & 5 & 8 & 0 \\ 3 & 9 & 6 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -6 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5

x_2 and x_4 are free variables
Let $x_2 = s$ and $x_4 = t$.

$x_3 + 4t = 0$.
$\underline{x_3 = -4t}$

$x_1 + 3s + 6t = 0$.
$\underline{x_1 = -(3s + 6t)}$

The null space of the matrix is

$V = \left\{ (x, y, z, \omega) \in R^4 \;\middle|\; \begin{pmatrix} -(3s + 6t) \\ s \\ -4t \\ t \end{pmatrix} \right\}$

5) **(15 points)** Let V be the set of all positive real numbers $\left(V = R^{++}\right)$ under the following operations:

Addition: $x+y$ is defined to be the product of the numbers x and y.
That is: $x+y = xy$ (For example: $2+3 = 2\times 3 = 6$)

Scalar Multiplication: kx is defined to be x raised to the power k, for any scalar $k \in R$.
That is: $kx = x^k$ (For example: $2x = x^2$ or $2.3 = 3^2 = 9$)

Show that V is a vector space.

① Let $\vec{u} = u \in V$ $\quad \vec{v} = v \in V$

$\vec{u} + \vec{v} = u + v = u \times v \in V$

② $\vec{u} + \vec{v} = u + v = u \times v$
$\vec{v} + \vec{u} = v + u = v \times u$ } COMMUTATIVITY

③ $(\vec{u}+\vec{v})+\vec{w} = (u+v)+w = (u\times v)+w = u\times v\times w$

$\vec{u}+(\vec{v}+\vec{w}) = u+(vw) = u\times v\times w = u\times v\times w = (\vec{u}+\vec{v})+\vec{w}$

∴ ASSOCIATIVITY

④ $\vec{0}+(\vec{u}) = \vec{u}$

$x_1 + u = u$

$x_1 u = u$

$x_1 = \frac{1}{u^2}$

$\vec{0} = \frac{1}{u^2} \in V$ and $\vec{0} + \vec{u} = \vec{u} = \vec{u} + \vec{0}$ since we showed that the addition is commutative.

⑤ $\vec{u} + (-\vec{u}) = \vec{0}$

$u + x_2 = \cancel{\frac{1}{u^2}}$

$u x_2 = \frac{1}{u^2}$

$x_2 = \frac{1}{u^3}$

$x_2 \in V$

⇒ There exists $(-\vec{u}) = \frac{1}{u^3} \in V$ with $\vec{u} + (-\vec{u}) = \vec{0} = (-\vec{u}) + \vec{u}$ (addition is commutative)

⑥ $\vec{u} \in V$ $\quad k\vec{u} = u^k \in V$

⑦ $k(\vec{u}+\vec{v}) = k(uv) = (uv)^k$

$k\vec{u} + k\vec{v} = u^k + v^k = (u^k)(v^k) = (uv)^k$