1) a) Determine the values of k for which the following matrix has an inverse:

$$A=\begin{pmatrix}0 & -2 & -k\\ 1 & 1 & 1\\ 1 & k & 3\end{pmatrix}. \qquad \text{(10 pts)}$$

b) Find the inverse of A for the special case when $k=1$. (10 pts)

10

1) a) $\begin{pmatrix}0 & -2 & -K\\ 1 & 1 & 1\\ 1 & K & 3\end{pmatrix} \xrightarrow{\text{int. } r_1 \text{ and } r_2} \begin{pmatrix}1 & 1 & 1\\ 0 & -2 & -K\\ 1 & K & 3\end{pmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & \frac{K}{2}\\ 1 & K & 3\end{pmatrix}$

$\xrightarrow{-r_1 \text{ to } r_3} \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & \frac{K}{2}\\ 0 & K-1 & 2\end{pmatrix}$

$\cancel{\frac{-1}{K-1}r_3}$

$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & \frac{K}{2}\\ 0 & 1 & \frac{2}{K-1}\end{pmatrix}$

$\boxed{2^{nd} \text{ Page}}$

$-r_2 \text{ to } r_3$

$\xrightarrow{-r_2 \text{ to } r_3} \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & \frac{K}{2}\\ 0 & 0 & \frac{K+4-K^2}{2(K-1)}\end{pmatrix}$

$K+4-K^2 = 2(K-1)$

$-K^2+K+4 = 2K-2$

$-K^2-K+6=0$

$K^2+K-6=0$

$K=2$ and $K=-3$

since $\Delta = 25$ and

$x_1 = \frac{-1-5}{2} = -3$

$x_2 = \frac{-1+5}{2} = 2$

$\xrightarrow{\frac{2(K-1)}{K+4-K^2}r_3} \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & \frac{K}{2}\\ 0 & 0 & 1\end{pmatrix}$

$\xrightarrow{-r_2 \text{ to } r_1} \begin{pmatrix}1 & 0 & 1-\frac{K}{2}\\ 0 & 1 & \frac{K}{2}\\ 0 & 0 & 1\end{pmatrix} \xrightarrow{-\frac{K}{2}r_3 \text{ to } r_2} \begin{pmatrix}1 & 0 & 1-\frac{K}{2}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$\xrightarrow{-(1-\frac{K}{2})r_3 \text{ to } r_1} \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$\cancel{\text{for}}$ $K+4-K^2 \neq 0$

$K^2-K-4 \neq 0$

$\Delta = 1-4(1)(-4) = 17$

$K_1 \neq \frac{1-\sqrt{17}}{2}$ $K_2 \neq \frac{1+\sqrt{17}}{2}$

for: $K \neq 1$

for $K \in \mathbb{R} - \left\{ \frac{1-\sqrt{17}}{2}, 1, \frac{1+\sqrt{17}}{2} \right\}$ A has An inverse

since $A \to I$.

b) $\begin{pmatrix} 0 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{\text{int } r_1 \text{ and } r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow[-r_1 \text{ to } r_3]{-\frac{1}{2} r_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{\frac{1}{2} r_3}$

10

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-r_2 \text{ to } r_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-\frac{1}{2} r_3 \text{ to } r_2]{-\frac{1}{2} r_3 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{int. } r_1 \text{ and } r_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-r_1 \text{ to } r_3]{-\frac{1}{2} r_2} \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2} r_3} \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\xrightarrow{-r_3 \text{ to } r_1} \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[-\frac{1}{2} r_3 \text{ to } r_2]{-\frac{1}{2} r_3 \text{ to } r_1} \begin{pmatrix} \frac{1}{2} & \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = A^{-1}$

d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{K}{2} \\ 0 & K-1 & 2 \end{pmatrix} \xrightarrow{-(K-1) r_2 \text{ to } r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{K}{2} \\ 0 & 0 & \frac{K+4-K^2}{2} \end{pmatrix} \xrightarrow{\frac{2}{-K^2+4+K} r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{K}{2} \\ 0 & 0 & 1 \end{pmatrix}$,

$\xrightarrow{-r_2 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 1-\frac{K}{2} \\ 0 & 1 & \frac{K}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{+\left(\frac{K-2}{2}\right) r_3 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{K}{2} \\ 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow{-\frac{K}{2} r_3 \text{ to } r_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$. since $\operatorname{rref}(A) = I \Rightarrow \exists A^{-1}$

But $K + 4 - K^2 \neq 0$.

$\therefore K^2 - K - 4 \neq 0$

$\Delta = (1) - 4(1)(-4) = 17 > 0$.

$K_1 = \frac{1-\sqrt{17}}{2} \qquad K_2 = \frac{1+\sqrt{17}}{2}$

for $K_1 \neq \frac{1-\sqrt{17}}{2}$ and $K \neq \frac{1+\sqrt{17}}{2}$ then $\exists A^{-1}$

2) a) Show that if AA^T (where A^T is the transpose of A) has no inverse, then A itself cannot have an inverse. (5 pts)

b) Show that if a matrix A is row equivalent to a matrix B, then there exists an invertible matrix Q such that $QA = B$ (and $Q^{-1}B = A$). (10 pts)

c) Show that if A can be written as the product of elementary matrices, then the linear system $Ax = 0$ has only the trivial solution. (5 pts)

a) Assume AA^T has an inverse → It is given with no inverse

$$(AA^T).(AA^T)^{-1} = I.$$

$$AA^T.(A^T)^{-1}.A^{-1} = I.$$

$A^T(A^T)^{-1} = I$ and $A.A^{-1} = I$ then $I = I$

thus AA^T has an inverse if and only if A has an inverse

and since AA^T has no ~~inf~~ inverse then A has no inverse.

b) A is row equivalent to Matrix B =>

$$A \xrightarrow{e_1} \cdots \xrightarrow{e_2} \cdots \xrightarrow{e_3} \cdots \xrightarrow{e_r} B.$$

then $E_r \ldots E_3 E_2 E_1 A = B.$

where $Q = E_r \ldots E_3 E_2 E_1$

and since Each elementary Matrix is invertible then Q is invertible since Q is A product of invertible Matrices

$$Q^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_r^{-1}$$

and ~~$E_r \ldots E_3 E_2$~~ ~~$E_1^{-1}E_2^{-1}E_3^{-1} \ldots E_r^{-1} E_r \ldots E_3 E_2 E_1$~~

$$E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_r^{-1} . E_r \ldots E_3 E_2 E_1 A = E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_r^{-1} B$$

$$IA = Q^{-1} B.$$

10

) If A can be written as a form of elementary Matrix => $\exists A^{-1}$ (property).

Since Elementary Matrices are all invertible
& product of e.m. is also invertible.

4

$$Ax = 0$$

$$A^{-1}Ax = A^{-1}0 = 0$$

$$Ix = 0$$

$$X = 0 \text{ (trivial solution)}$$ unique

prove.

3) A matrix A is said to be an **idempotent** matrix if $A^2 = A$. Prove the following

a) If A is idempotent, then $I - A$ is also idempotent. **(5 pts)**
b) If A is idempotent and invertible then $A = I$. **(5 pts)**
c) If A is idempotent then $I - 2A$ is invertible. Find $(I-2A)^{-1}$ in terms of A. (5 pts)
d) Give an example of a 2×2 idempotent matrix A such that A is not the zero or the identity matrix. (5 pts)

a) $A^2 = A \Rightarrow (I-A)^2 = I - A$.

$(I-A)^2 = I.I + A.A - 2AI = I + A - 2A$

$= I + A - 2A = I - A$.

5

Since $A.A = A^2 = A$ since A is idempotent.

and $A.I = IA = A$

b) $A^2 = A$ and $\exists A^{-1} \Rightarrow A = I$

$A^2 = A$.

$A^{-1}.A.A = A^{-1}.A$.

5

$I.A = I \Rightarrow I = A$

c) ~~Since $I - 2A$ is invertible $\Rightarrow$~~

$\cancel{(I-2A) \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_3} I}$

$A^2 = A$ (Since A is said To be idempotent)

$-2A^2 = -2A$.

$I - 2A^2 = I - 2A$.

$A_2 I_2 = 2A$)

d) let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$A \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A$$

5

then $A^2 = A$.

a) Find a square matrix A that satisfies $A^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. (10 pts)

b) Let A be a square matrix. Prove that if the homogeneous system $Ax = 0$ has a unique solution (the trivial solution), then the system $A^k x = 0$ also has a unique solution. (10 pts)

a) A is a diagonal matrix since it is a 3×3 matrix and all the entries besides on the diagonal are zeros then $A^5 = \begin{bmatrix} (1)^5 & 0 & 0 \\ 0 & (-1)^5 & 0 \\ 0 & 0 & (-1)^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ 0 & -1 & \\ 0 & 0 & - \end{bmatrix}$

10

then $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & & -1 \end{pmatrix}$

b)

$Ax = 0$ has unique Sol. the trival sol $\Rightarrow$ $A^k x = 0$ has a unique Sol.

IF $Ax = 0$ has unique Solution The trival Solution then $\exists A^{-1}$ (property).

$A^{-1} A x = 0$

$x = 0$ (unique trival solution)

7

then
$$A^k x = 0$$
$$A^{-1} A^k x = A^{-1} 0 . 0$$
$$A^{k-1} x = 0 \quad \dots \text{ After } k \text{ times}$$
$$(A^{-1})^k . A^k x = (A^{-1})^k 0 = 0$$
$$(A^k)^{-1} . A^k x = 0 .$$
$$I x = 0$$
$x = 0$ (then the system $A^k x = 0$ has a unique solution which is the Trival Solution)

prove uniqueness//

$(A^{-1})^k = (A^k)^{-1}$

Since $(\underbrace{A . A . A . A \dots}_{K \text{ times}})^{-1} = (\underbrace{A^{-1} . A^{-1} \dots}_{K \text{ times}})$
$= (A^{-1})^k$

5) Solve the following linear system of equations by Gauss-Jordan elimination:

$$
\begin{aligned}
2x_1 + 2x_2 + 10x_3 &= -2 \\
x_1 + 2x_2 + 7x_3 + x_4 + 4x_6 &= -3 \\
x_1 + x_2 + 5x_3 + x_4 - x_6 &= 0 \\
2x_1 + 2x_2 + 10x_3 + 2x_4 + x_5 &= 2
\end{aligned}
$$

(20 pts)

20

$$
\begin{pmatrix} 2 & 2 & 10 & 0 & 0 & 0 & -2 \\ 1 & 2 & 7 & 1 & 0 & 4 & -3 \\ 1 & 1 & 5 & 1 & 0 & -1 & 0 \\ 2 & 2 & 10 & 2 & 1 & 0 & 2 \end{pmatrix}
\xrightarrow[-r_3 \text{ to } r_2]{\frac{1}{2} r_1}
$$

$$
\begin{pmatrix} 1 & 1 & 5 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 5 & -3 \\ 1 & 1 & 5 & 1 & 0 & -1 & 0 \\ 2 & 2 & 10 & 2 & 1 & 0 & 2 \end{pmatrix}
\xrightarrow[-2r_1 \text{ to } r_4]{-r_1 \text{ to } r_3}
\begin{pmatrix} 1 & 1 & 5 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 4 \end{pmatrix}
$$

$$
\xrightarrow{-2r_3 \text{ to } r_4}
\begin{pmatrix} 1 & 1 & 5 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}
\xrightarrow{-r_2 \text{ to } r_1}
\begin{pmatrix} 1 & 0 & 3 & 0 & 0 & -5 & \\ 0 & 1 & 2 & 0 & 0 & 5 & \\ 0 & 0 & 0 & 1 & 0 & -1 & \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}
$$

let $x_6 = t$

$x_5 = 2 - 2t$

$x_4 = 1 + t$

let $x_3 = s$

$x_2 = -3 - 5t - 2s$

$x_1 = 2 + 5t - 3s$