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NDU

MAT 215

Linear Algebra I

Exam # 1

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Duration: 75 minutes

Name: Georges Akiki

Section: A

Instructor: Maalouf

Grade: ____________

Problem Number	Points	Score
1	18	
2	14	
3	7	
4	13	
5	12	
6	16	
7	20	
Total	100	

1) **(18 points)** Use Gauss-Jordan elimination to solve the following system of equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 \qquad\quad + x_5 + x_6 &= 4 \\
2x_1 + 4x_2 + 6x_3 - 2x_4 + 2x_5 &= 6 \\
3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 + 4x_6 &= 15 \\
4x_1 + 4x_2 + 4x_3 + x_4 + 5x_5 + 3x_6 &= 20 \\
x_1 + 3x_2 + 5x_3 - 2x_4 + x_5 - x_6 &= 2
\end{aligned}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
2 & 4 & 6 & -2 & 2 & 0 & 6 \\
3 & 2 & 1 & 2 & 3 & 4 & 15 \\
4 & 4 & 4 & 1 & 5 & 3 & 20 \\
1 & 3 & 5 & -2 & 1 & -1 & 2
\end{pmatrix}
\xrightarrow[\substack{-3r_1 \text{ to } r_3 \\ -4r_1 \text{ to } r_4 \\ -r_1 \text{ to } r_5}]{-2r_1 \text{ to } r_2}
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 2 & 4 & -2 & 0 & -2 & -2 \\
0 & -1 & -2 & 2 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 2 & 4 & -2 & 0 & -2 & -2
\end{pmatrix}
\xrightarrow[\frac{1}{2} r_5]{\frac{1}{2} r_2}
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 1 & 2 & -1 & 0 & -1 & -1 \\
0 & -1 & -2 & 2 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 1 & 2 & -1 & 0 & -1 & -1
\end{pmatrix}
\xrightarrow[\substack{r_2 \text{ to } r_3 \\ -r_2 \text{ to } r_5}]{-r_2 \text{ to } r_1}
\begin{pmatrix}
1 & 0 & -1 & 1 & 1 & 2 & 5 \\
0 & 1 & 2 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow[\substack{r_3 \text{ to } r_2 \\ -r_3 \text{ to } r_4}]{-r_3 \text{ to } r_1}
$$

$$
\begin{pmatrix}
1 & 0 & -1 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{-r_4 \text{ to } r_1}
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 3 & 1 \\
0 & 1 & 2 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$r<6$

Let $x_6 = t$ $\quad x_5 = 2+t$ $\quad x_4 = 2$

Let $x_3 = s$ $\quad x_2 = 1+t-2s$ $\quad x_1 = 1-3t+s$

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2) (14 points) Discuss, according to the values of k, the nature of the solutions of the system whose augmented matrix is below. Then, solve the system whenever possible.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & k^2 & k \end{array}\right]$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & k^2 & k \end{array}\right) \xrightarrow[-r_1 \text{ to } r_3]{-r_1 \text{ to } r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right) \xrightarrow{1/2 r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right)$$

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for $k=-1$ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right)$ no solution

for $k=1$ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ infinitely many solutions ← find solution

for $k \neq \pm 1$ $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right)$

$X_3 = \frac{k-1}{k^2-1} = \frac{k-1}{(k-1)(k+1)}$ $\boxed{X_3 = \frac{1}{k+1}}$

$X_2 = -1 - \frac{1}{k+1} = \frac{-k-1-1}{k+1}$ $\boxed{X_2 = \frac{-k-2}{k+1}}$

$X_1 = 1 - \frac{1}{k+1} = \frac{k+1-1}{k+1}$ $\boxed{X_1 = \frac{k}{k+1}}$

3) **(7 points)** Find the matrix A, given that $\left(2A^T+I\right)^{-1}=\begin{pmatrix}4 & 5\\ 6 & 8\end{pmatrix}$

$$(2A^t+I)^{-1}=\begin{pmatrix}4 & 5\\ 6 & 8\end{pmatrix}$$

$$\left[(2A^t+I)^{-1}\right]^{-1}=\frac{1}{2}\begin{pmatrix}8 & -5\\ -6 & 4\end{pmatrix}=\begin{pmatrix}4 & -5/2\\ -3 & 2\end{pmatrix}$$

$$2A^t+I=\begin{pmatrix}4 & -5/2\\ -3 & 2\end{pmatrix}$$

$$2A^t=\begin{pmatrix}4 & -5/2\\ -3 & 2\end{pmatrix}-I$$

$$2A^t=\begin{pmatrix}4 & -5/2\\ -3 & 2\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}3 & -5/2\\ -3 & 1\end{pmatrix}$$

$$2A^t=\begin{pmatrix}3 & -5/2\\ -3 & 1\end{pmatrix}$$

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$$A^t=\frac{1}{2}\begin{pmatrix}3 & -5/2\\ -3 & 1\end{pmatrix}=\begin{pmatrix}3/2 & -5/4\\ -3/2 & 1/2\end{pmatrix}$$

$$(A^t)^t=\begin{pmatrix}3/2 & -5/4\\ -3/2 & 1/2\end{pmatrix}^t=\begin{pmatrix}3/2 & -3/2\\ -5/4 & 1/2\end{pmatrix}$$

$$A=\begin{pmatrix}3/2 & -3/2\\ -5/4 & 1/2\end{pmatrix}$$

4) **(13 points)**

a) **(8 points)** Find the inverse of the matrix $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$.

b) **(5 points)** Use the result of part (a) to solve the matrix equation
$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{pmatrix}.$$

a) $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \xrightarrow{-r_1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \xrightarrow[-3r_1 \text{ to } r_3]{-r_1 \text{ to } r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -4 \end{pmatrix} \xrightarrow{-r_2 \text{ to } r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix}$

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$\xrightarrow{-1/3\, r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \text{ to } r_2]{-r_3 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

calculation mistakes

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-r_1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-3r_1 \text{ to } r_3]{-r_1 \text{ to } r_2} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow{-r_2 \text{ to } r_3} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow{-1/3\, r_3} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -2/3 & 1/3 & -1/3 \end{pmatrix}$

$\xrightarrow[r_3 \text{ to } r_2]{-r_3 \text{ to } r_1} \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/3 & 4/3 & -1/3 \\ -2/3 & 1/3 & -1/3 \end{pmatrix} = A^{-1} \quad A^{-1} \neq \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$

b) $AX = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{pmatrix} \qquad A^{-1}AX = A^{-1}\begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{pmatrix} \qquad IX = A^{-1}\begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{pmatrix} \qquad X = A^{-1}\begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{pmatrix}$

$X = \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/3 & 4/3 & -1/3 \\ -2/3 & 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ -11/3 & 1/3 \\ -5/3 & -8/3 \end{pmatrix}$

$X = \begin{pmatrix} 2/3 & 2/3 \\ -11/3 & 1/3 \\ -5/3 & -8/3 \end{pmatrix}$

o.k.

5) **(12 points)** Consider the matrices $A=\begin{bmatrix}4&5&6\\1&2&3\\1&1&1\end{bmatrix}$ and $B=\begin{bmatrix}1&2&3\\4&5&6\\3&5&7\end{bmatrix}$.

a) **(6 points)** Find elementary matrices E_1 and E_2 such that $B=E_2E_1A$.

b) **(6 points)** Use part (a) to find elementary matrices G and F such that $A=GFB$.

a) $A=\begin{bmatrix}4&5&6\\1&2&3\\1&1&1\end{bmatrix} \xrightarrow{\text{interchange } r_1 \, r_2} \begin{bmatrix}1&2&3\\4&5&6\\1&1&1\end{bmatrix} \xrightarrow{2r_1 \text{ to } r_3} \begin{bmatrix}1&2&3\\4&5&6\\3&5&7\end{bmatrix}=B$

$I_3\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\text{interchange } r_1 r_2} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=E_1$

$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{2r_1 \text{ to } r_3} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}=E_2$

~~Th[illegible]~~ $\therefore\ B=E_2E_1A$

b) $E_2^{-1}B=E_1A \qquad E_1^{-1}E_2^{-1}B=A$

$F=E_2^{-1}$ and $G=E_1^{-1}$

$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{\text{interchange } r_1 r_2} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=E_1^{-1}=G$ the inverse of an interchange is interchanging the same rows

$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{-2r_1 \text{ to } r_3} \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}=E_2^{-1}=F$ The inverse of adding twice the row$_1$ to row$_3$ is subtracting twice row$_1$ from row$_3$

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6) (16 points) Prove the following theorem:

If A is an $n \times n$ matrix, then the following statements are equivalent:
a) A is invertible.
b) $A\bar{x} = \bar{o}$ has only the trivial solution.
c) The reduced row-echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.

Hint: Prove that (a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (d), (d) ⇒ (a).

If A is invertible then there exist 1 unique matrix A^{-1} such that $A^{-1}A = I$

b) $A\vec{X} = \vec{O}$ $A^{-1}A\vec{X} = A^{-1}\vec{O}$ $I\vec{X} = A^{-1}\vec{O}$ $\vec{X} = A^{-1}\vec{O}$ $\vec{X} = \vec{O}$

any matrix multiplied by $\vec{O}$ gives $\vec{O}$

a) ⇒ b) ∴ $A\vec{X} = \vec{O}$ has only the trivial solution $\vec{O}$

c) $A\vec{X} = O$ Augmented matrix $[A \mid O]$

since there exist only 1 solution which is the trivial solution we can straight

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

no rows ~~can~~ be a zero row since we would have had ∞ many solutions

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ & & & a_{nn} \end{pmatrix} \xrightarrow{e_1} \xrightarrow{e_2} \xrightarrow{e_m} I_n$$ since the r.r.e.f of A is I_n

Let E_k be the elementary matrix which represents e_k

d) so $E_m \cdots E_2 E_1 A = I$

$E_m^{-1} E_m \cdots E_2 E_1 A = E_m^{-1} I \iff E_{m-1} \cdots E_2 E_1 A = E_m^{-1} I$

$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_m^{-1} I$

$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_m^{-1}$

& $E_1^{-1}, E_2^{-1} \cdots E_m^{-1}$ are elementary matrices.

So A can be expressed as a product of elementary matrices

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7) **(20 points)** Determine whether each of the following statements is true or false. Justify your answer.

a) A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions. True

homogeneous systems always has the trivial solution
When wrighting the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \end{array}\right]$$

Always we can take x_4 = to a variable t and we can get x_1, x_2 & x_3 as function of t. It is impossible to have no solution ~~since~~ we can never have a row $(0\ 0\ 0\ 0 \mid K)$

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b) If $AB = AC$, then $B = C$. False

~~multiply both side by A^{-1}~~ ~~$AB = AC$~~ ~~$A^{-1}A = I$~~ ~~$A^{-1}AB = A^{-1}AC$~~ ~~$\Leftrightarrow B = C$~~

Only if A is ~~square~~ invertible then we can wright

$A^{-1}AB = A^{-1}AC$
$IB = IC$
$B = C$

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c) If A and B are invertible $n\times n$ matrices, then the matrix AB is invertible. True

$(AB)^{-1} = (B^{-1}A^{-1})(AB) = I$

Since A & B are invertible so there exist B^{-1} & A^{-1}

So AB is invertible.

prove it

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d) If A and B are symmetric $n \times n$ matrices, then the matrix AB is symmetric. False

A & B are symmetric. So $A^t = A$ $B^t = B$

$(AB)^t = B^t A^t = BA \neq AB$

~~Since A & B are symmetric~~ $(AB)^t = BA$ to be symmetric $(AB)^t = AB$

~~Row: both are entries as columns in A & B~~

~~If we multiply the i^{th} row in A by the column in B is the same as multiplying the j^{th} row in B by the i^{th} column A. So BA & AB will have the same entries~~

~~So $(AB)^t = AB$~~ ~~So AB is symmetric~~ ~~So BA = AB~~

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e) If A is an $n \times n$ matrix which has a row of zeros, then A is not invertible. True

Let $AB = C$

the i^{th} row in C is the i^{th} row in $A \times$ columns of B

If the i^{th} row of A was all zeros then C will also have a zero row

$\therefore$ C can never be the identity matrix.

So there is no matrix such that $AB = I$. So A is not invertible

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