NDU

MAT 215

Linear Algebra I

Exam # 1

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Duration: 75 minutes

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Section: MWF 10→11 B

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Grade: ____________

Problem Number	Points	Score
1	18	17
2	14	$13\frac{1}{2}$
3	7	7
4	13	13
5	12	10
6	16	16
7	20	12
Total	100	89

1) (18 points) Use Gauss-Jordan elimination to solve the following system of equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 \qquad\quad + x_5 + x_6 &= 4 \\
2x_1 + 4x_2 + 6x_3 - 2x_4 + 2x_5 &= 6 \\
3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 + 4x_6 &= 15 \\
4x_1 + 4x_2 + 4x_3 + x_4 + 5x_5 + 3x_6 &= 20 \\
x_1 + 3x_2 + 5x_3 - 2x_4 + x_5 - x_6 &= 2
\end{aligned}
$$

$$
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
2 & 4 & 6 & -2 & 2 & 0 & 6 \\
3 & 2 & 1 & 2 & 3 & 4 & 15 \\
4 & 4 & 4 & 1 & 5 & 3 & 20 \\
1 & 3 & 5 & -2 & 1 & -1 & 2
\end{array}\right]
\quad \times(-2),\ \times(-3),\ \times(-4),\ \times(-1)
$$

$$
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 2 & 4 & -2 & 0 & -2 & -2 \\
0 & -1 & -2 & 2 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 2 & 4 & -2 & 0 & -2 & -2
\end{array}\right]
\times\tfrac{1}{2}
\quad
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 1 & 2 & -1 & 0 & -1 & -1 \\
0 & -1 & -2 & 2 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 2 & 4 & -2 & 0 & -2 & -2
\end{array}\right]
$$

$$
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 1 & 2 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\quad
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 1 & -2 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\quad
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 1 & 4 \\
0 & 1 & -2 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\left[\begin{array}{cccccc|c}
1 & 1 & 0 & 0 & 1 & 1 & 5 \\
0 & 1 & 0 & 0 & 1 & -2 & 4 \\
0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\times(-1)
\quad
\left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 3 & 1 \\
0 & 1 & 0 & 0 & 1 & -2 & 4 \\
0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 1 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$\Rightarrow$ x_5 and x_6 are free variables let $x_5 = t$ and $x_6 = s$

$$
\begin{aligned}
x_1 + 3x_6 &= 1 \\
x_2 + x_5 - 2x_6 &= 4 \\
x_3 &= -\tfrac{1}{2} \\
x_4 + x_5 - x_6 &= 4.
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
x_1 &= -3s + 1 \\
x_2 &= -t + 2s + 4 \\
x_3 &= -\tfrac{1}{2} \\
x_4 &= -t + s + 4
\end{aligned}
$$

~~the solution of the system~~

$$\cancel{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3s+1 \\ -t+2s+4 \\ -\frac{1}{2} \\ -t+s+4 \\ t \\ s \end{pmatrix}}$$

following:

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & -2 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ ok.}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & -2 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ ok.} \quad \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \hline 1 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \text{ ok.} \quad \overset{\text{ok.}}{\left[\begin{array}{cccccc|c} 1 & 0 & 3 & 0 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}$$

we have x_3 and x_6 free variables

let $x_3 = t$ and $x_6 = s$.

following:

$x_1 + 3x_3 + 3x_6 = 1 \implies x_1 = -3t - 3s + 1$

$x_2 - 2x_3 - x_6 = 1 \qquad x_2 = 2t + s + 1$

$x_4 = 2. \qquad x_4 = 2$

$x_5 - x_6 = 2. \qquad x_5 = s + 2.$

solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3t-3s+1 \\ 2t+s+1 \\ t \\ 2 \\ s+2 \\ s \end{pmatrix} \begin{matrix} \text{ok.} \\ \text{ok.} \\ \\ \\ \\ \\ \end{matrix}$$

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2) (14 points) Discuss, according to the values of k, the nature of the solutions of the system whose augmented matrix is below. Then, solve the system whenever possible.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & k^2 & k \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & k^2 & k \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right]$$

$k^2-1=0$

$\Rightarrow (k-1)(k+1)=0$

$\Rightarrow k=\pm 1$

if $k=+1$ the system becomes $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

the system has infinitely many solution

let $x_3=t$ free variable)

$x_1=-t+1$

$x_2=-t-1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t+1 \\ -t-1 \\ t \end{pmatrix}$$

if $k=-1$. the system becons $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 \end{array}\right]$ the system is inconsistent it has no solutions

because we have $0x_1+0x_2+0x_3=-2$

if $k\neq\pm 1$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right] \times\frac{1}{2}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right] \times\frac{1}{(k^2-1)} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & \frac{k-1}{k^2-1} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{k+1} \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{k+1}+1 \\ 0 & 1 & 0 & -\frac{1}{k+1}-1 \\ 0 & 0 & 1 & \frac{1}{k+1} \end{array}\right]$$

the system has a unique solution for $k\neq\pm 1$

$13\frac{1}{2}$

3) **(7 points)** Find the matrix A, given that $\left(2A^T+I\right)^{-1} = \begin{pmatrix} 4 & 5 \\ 6 & 8 \end{pmatrix}$

$$(2A^T+I)^{-1} = \begin{bmatrix} 4 & 5 \\ 6 & 8 \end{bmatrix} \quad ad-bc = 32-30 = 2 \neq 0$$

the matrix is invertible

$$\cancel{(2A^T+I)} \quad \left[(2A^T+I)^{-1}\right]^{-1} = (2A^T+I) = \frac{1}{2}\begin{bmatrix} 8 & -5 \\ -6 & 4 \end{bmatrix}$$

$$\Rightarrow (2A^T+I) = \begin{bmatrix} 4 & -\frac{5}{2} \\ -3 & 2 \end{bmatrix} \Rightarrow 2A^T = \begin{bmatrix} 4 & -\frac{5}{2} \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2A^T = \begin{bmatrix} 3 & -\frac{5}{2} \\ -3 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \frac{3}{2} & -\frac{5}{4} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{5}{4} & \frac{1}{2} \end{bmatrix}$$

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4) (13 points)

a) **(8 points)** Find the inverse of the matrix $A=\begin{pmatrix}-1 & 0 & 1\\ 1 & 1 & 0\\ 3 & 1 & -1\end{pmatrix}$.

b) **(5 points)** Use the result of part (a) to solve the matrix equation

$$\begin{pmatrix}-1 & 0 & 1\\ 1 & 1 & 0\\ 3 & 1 & -1\end{pmatrix}\mathbf{X}=\begin{pmatrix}1 & 2\\ -3 & 1\\ 0 & 5\end{pmatrix}.$$

a) $\left[\begin{array}{ccc|ccc}-1 & 0 & 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 3 & 1 & -1 & 0 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & -1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 3 & 1 & -1 & 0 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & -1 & -1 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 2 & 3 & 0 & 1\end{array}\right]$

$\left[\begin{array}{ccc|ccc}1 & 0 & -1 & -1 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 2 & -1 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 1\\ 0 & 1 & 0 & -1 & 2 & -1\\ 0 & 0 & 1 & 2 & -1 & 1\end{array}\right].$

$\Rightarrow A^{-1}=\begin{bmatrix}1 & -1 & 1\\ -1 & 2 & -1\\ 2 & -1 & 1\end{bmatrix}$

b) $AX=B \Rightarrow X=A^{-1}B$

$X=\begin{bmatrix}1 & -1 & 1\\ -1 & 2 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & 1\\ 0 & 5\end{bmatrix}$

$X=\begin{bmatrix}4 & 6\\ -7 & -5\\ 5 & 8\end{bmatrix}$

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5) **(12 points)** Consider the matrices $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

a) **(6 points)** Find elementary matrices E_1 and E_2 such that $B = E_2 E_1 A$.

b) **(6 points)** Use part (a) to find elementary matrices G and F such that $A = GFB$.

a) $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

$B = E_2 E_1 A$ to change A to B we have to interchange rows 1 and 2 and then add 2 times row 2 to row 3.

$A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow E_2 A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$

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$E_2 E_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 5 & 7 \end{bmatrix} = B.$

b) follow:
$A = GFB$. $B = E_2 E_1 A$.

$(E_2 E_1)^{-1} B = (E_2 E_1)^{-1} (E_2 E_1) A$

$(E_2 E_1)^{-1} B = IA$.

$E_1^{-1} E_2^{-1} B = A$.

6

and $GFB = A$

$\Rightarrow G = E_1^{-1}$ and $F = E_2^{-1}$ $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

$G = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

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6) **(16 points)** Prove the following theorem:

If A is an $n \times n$ matrix, then the following statements are equivalent:
a) A is invertible.
b) $A\vec{x} = \vec{o}$ has only the trivial solution.
c) The reduced row-echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.

Hint: Prove that (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (a).

Given: A $n \times n$ matrix.
RTP: A invertible, $A\vec{x} = \vec{0}$ has only the trivial solution, The reduced row echelon form of A is I_n, A is expressible as a product of elementary matrices.

Proof: $a \Rightarrow b$. A invertible

$$A\vec{x} = \vec{0} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{0}$$
$$\Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}.$$

$A\vec{x} = \vec{0}$ has only the trivial solution

$b \Rightarrow c$ If $A\vec{x} = \vec{0}$ has only the trivial solution and since A is an $n \times n$ matrix $\Rightarrow$ A has n leading 1's, and no row of zero
then then ~~A is~~ $\cancel{A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}}$ $A = \left[\begin{array}{cccc|c} 1 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$

$\Rightarrow$ the reduced row echelon form of A is I_n.

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$c \Rightarrow d$. Reduced row echelon form is I_n. $\Rightarrow$ ~~A~~ there are $E_1 E_2 \cdots E_k$ elementary matrices that transfers A to I $\Rightarrow (E_k \cdots E_2 E_1) A = I$ (row operations) and the elementary matrices are invertible and their inverses are elementary matrices $(E_k \cdots E_2 E_1)^{-1}(E_k \cdots E_2 E_1) A = (E_k \cdots E_2 E_1)^{-1} I$
$\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I$ $\Rightarrow$ A is the product of elementary matrices

$d \Rightarrow a$ A is the product of elementary matrices. $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I$
elementary matrices are invertible I is invertible $\Rightarrow$ the product of elementary matrices is invertible $\Rightarrow$ A is invertible

7) (20 points) Determine whether each of the following statements is true or false. Justify your answer.

a) A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

~~[illegible]~~ True / Because suppose the system is represented by the ~~l~~ augmented matrix $A_{m \times n}$ ~~[illegible]~~ $m < n$.
the maximum we can have is ~~m~~ leading 1's and $m <$
this means there still are free variables $\Rightarrow$ the system will have infinitely many solutions

Also, homogeneous $\Rightarrow$ consistent

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b) If $AB = AC$, then $B = C$.

False
Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $AB = AC$

but $B \neq C$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

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c) If A and B are invertible $n \times n$ matrices, then the matrix AB is invertible.

True $\frac{1}{2}$

A invertible and B invertible and Both squared $n \times n$ matrices $\Rightarrow$ AB invertible
because the product of invertible matrices is invertible

Why?? This is what you have to prove

d) If A and B are symmetric $n \times n$ matrices, then the matrix AB is symmetric.

~~True~~. No!. $-\frac{1}{2}$

e) If A is an $n \times n$ matrix which has a row of zeros, then A is not invertible.

True.

A $m \times n$ matrix with a row of zeros

If A is invertible ~~the~~ consider B ~~...~~ its inverse

then $AB = BA = I$

But A has a row of zeros. for every $i = 1, \ldots, m$

$\Rightarrow$ i^{th} row of $AB = i^{th}$ row of $A \times B$

~~[0 ... 0]~~ i^{th} row of $AB = [0 \ldots 0] \times B$

$= [0 \ldots 0]$.

for every i the ~~...~~ all the rows are zeros

$\Rightarrow AB = 0 \neq I$

$\Rightarrow A$ is not invertible

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