1) **(20 points)** For the matrix: $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & -k & -1 & 2 \\ 1 & 1 & k & 1 \\ 2 & 2 & -2 & k-1 \end{pmatrix}$.

a) **(14 points)** Find the determinant of A.

$$\det A = \begin{vmatrix} 1 & 1 & -1 & 2 \\ 1 & -K & -1 & 2 \\ 1 & 1 & K & 1 \\ 2 & 2 & -2 & K-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & 2 \\ 0 & -K-1 & 0 & 0 \\ 0 & 0 & K+1 & -1 \\ 0 & 0 & 0 & K-1-4 \end{vmatrix}$$

we did:
$R_2 = R_2 - R_1$
$R_3 = R_3 - R_1$
$R_4 = R_4 - 2R_1$

$$= \begin{vmatrix} 1 & 1 & -1 & 2 \\ 0 & -K-1 & 0 & 0 \\ 0 & 0 & K+1 & -1 \\ 0 & 0 & 0 & K-5 \end{vmatrix}$$

$$= 1 \cdot (-K-1)(K+1)(K-5)$$
$$= -(K+1)^2(K-5)$$

b) **(6 points)** Find the value(s) of k for which A is invertible.

A is invertible $\Leftrightarrow$ then $\det(A) \neq 0$

$$-(K+1)^2(K-5) \neq 0$$

~~K = -1 or K =~~

$K + 1 = 0 \Leftrightarrow \boxed{K = -1}$

$K - 5 = 0 \Leftrightarrow \boxed{K = 5}$

2) (14 points)

a) Show that Cramer's rule can be applied to solve the following system:

$$x + y + z = 2$$
$$x - y + z = 0$$
$$x - y - z = -2$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ $\quad \det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$

$= \big((-1)(-1) - (-1)(1)\big)(1) - \big((1)(1) - (1)(1)\big)(1) + \big((1)(-1) - (1)(-1)\big)(1)$

$= 1+1 - (-1 -1) + (-1) + 1 = 4 \neq 0$

$\det A = 4 \neq 0$ then cramer's rule can be applied

b) Solve only for x.

$A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ -2 & -1 & -1 \end{bmatrix}$ $\quad \det(A_1) = 2 \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} + (-2) \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$

$= 2(1+1) + (-2)(1+1)$

$= 4 + (-4)$

$= 0$

$x = \dfrac{\det(A_1)}{\det(A)} = \dfrac{0}{4} = 0$

3) (10 points) Without directly evaluating, show that

$$\begin{vmatrix} a & 3 & b+c+d \\ b & 3 & a+c+d \\ c & 3 & b+a+d \end{vmatrix} = 0$$

-10

$\begin{vmatrix} 3 & b+c+d \\ 3 & a+c+d \\ 3 & b+a+d \end{vmatrix} = 3 \begin{vmatrix} a & 1 & b+c+d \\ b & 1 & a+c+d \\ c & 1 & b+a+d \end{vmatrix}$

$= 3\Big[-1\big[b(b+a+d) - c(a+c+d)\big] + 1\big[a(b+a+d) - c(b+c+d)\big]$
$\quad -1\big[a(a+c+d) - b(b+c+d)\big]\Big]$

$= 3(-b^2 - ab - bd + ac + c^2 + cd + ab + a^2 + ad - cb - c^2 - cd$
$\quad - a^2 - ac - ad + b^2 + bc + bd)$

$= (3) \cdot 0 = 0$

4) (36 points) Let A be an $n \times n$ matrix. Prove each of the following

a) If A is upper triangular, then $|A| = a_{11}a_{22}\dots a_{nn}$ by using the combinatorial approach.

A is upper triangular $\Longleftrightarrow$ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \Longleftrightarrow |A| = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix}$$

✓

$$= a_{11} \cdot a_{22} \cdot a_{33}$$

and so does if it is $m \times m$

$$|A| = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{m \times m}$$

b) If A is invertible, then $\det(adj(A)) = [\det(A)]^{n-1}$.

A invertible $\Longleftrightarrow$ $\det A \neq 0$ $\Longleftrightarrow$ $A^{-1} = \frac{1}{\det A} \cdot adj(A)$

$$adj(A) = \det A \cdot A^{-1}$$

$\det(A^{-1}) = \frac{1}{\det(A)}$

$$\det(adj(A)) = \det((\det A) \cdot A^{-1})$$

$$= \det(\det A) \cdot \det(A^{-1})$$

$$= |A|^m \cdot \cancel{|A|^{-1}}$$

✓

$$\det(adj(A)) = |A|^{m-1}$$

2

$$\Longleftrightarrow \det(adj(A)) = [\det(A)]^{m-1}$$

c) If $A^2 = A$, then $|A| = 0$ or $|A| = 1$.

$A^2 = A \Longleftrightarrow \det(A^2) = \det(A)$ ✓

~~det(A) det~~

$$\det(A \cdot A) = \det(A)$$

$$\det(A) \cdot \det(A) = \det(A)$$

$$|A| \cdot |A| - |A| = 0$$

$$|A|(|A| - 1) = 0$$

$\Rightarrow \boxed{|A| = 0}$ or $|A| - 1 = 0$

$\boxed{|A| = 1}$

d) If A is invertible, then A^4 cannot be equal to the zero matrix.

A is invertible $\iff$ det $A \neq 0$

det $A^4 = |A.A.A.A| = |A|.|A|.|A|.|A| \neq 0$ ✓

~~so det A^4~~ so A^4 cannot be equal to the zero matrix because a zero matrix has a determinant equal to 0.

e) If $|A| = 1$ and all entries in A are integers, then all entries in A^{-1} are integers.

Sure if $|A| = 1$ and all entries in A are integers, then all entries in A^{-1} are integers because the cofactors of A will be integers and the adj(A) will have all entries integers and det(A) is an integer ✓

and $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \text{adj}(A)$ in which all entries of adj(A) are integers.

f) If B is an $n \times n$ matrix, then $|AB| = |A||B|$.

$|AB| = |A|.|B|$

✓

2 cases:

① if A is not invertible, then det A = 0 and then det(A). det(B) = 0 $\iff$ so det(A) = det(B) = 0

and det(AB) = det(A). det B

$|AB| = |A|.|B|$

② if A is invertible, then det A $\neq 0$

✓

$A = \overbrace{E_1 . E_2 \cdots E_K}^{A}$

$\det(AB) = \det(E_1 . E_2 . E_3 \cdots E_K \cdot B)$

~~= det E~~ det

$= |E_1| . |E_2| \cdots |E_K| . |B|$

5) **(20 points)** Let $V = R^2$, the set of all pairs of real numbers, with the operations:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 1)$$
$$k \circ (x, y) = (kx, y)$$

Check all 10 axioms that must hold for V to be a vector space, and determine which axioms are true and which are false. Then determine whether or not V is a vector space.

AXIOM 1: $u + V = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 1)$

true ✓

it is a vector space.

AXIOM 2: $u + V \overset{?}{=} V + u$

$u + V$	$V + u$
$= (x_1, y_1) + (x_2, y_2)$	$= (x_2, y_2) + (x_1, y_1)$
$= (x_1 + x_2, y_1 + y_2 + 1)$	$= (x_2 + x_1, y_2 + y_1 + 1)$
	$= (x_1 + x_2, y_1 + y_2 + 1)$

✓

true

vector space.

AXIOM 3: $u + (V + w) \overset{?}{=} (u + V) + w$

$u + (V + w)$	$(u + V) + w$
$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$	$= ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$
$(x_1, y_1) + (x_2 + x_3, y_2 + y_3 + 1)$	$= (x_1 + x_2, y_1 + y_2 + 1) + (x_3, y_3)$
$(x_1 + x_2 + x_3, y_1 + y_2 + y_3 + 2)$	$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3 + 2)$

✓

true Vector space.

AXIOM 4 0 vector

Let $0(x_2, y_2)$

$0 + u = u$

$(x_2, y_2) + (x_1, y_1) = (x_1, y_1)$

$(x_2 + x_1, y_2 + y_1 + 1) = (x_1, y_1)$

✓

$\Rightarrow x_2 = 0$

$y_2 + 1 = 0 \Leftrightarrow y_2 = -1 \Leftrightarrow 0(0, -1)$

$(0, -1) + (x_1, y_1) = \cancel{(x_1, y_1)}\ (0 + x_1, y_1 - 1 + 1) = (x_1, y_1)$

AXIOM 5 : Let $-u (x_2, y_2)$

$-u + u = 0 \iff (x_2, y_2) + (x_1, y_1) = (0, -1)$

$(x_1 + x_2, y_1 + y_2 + 1) = (0, -1)$

$x_1 + x_2 = 0 \iff x_2 = -x_1$

$y_1 + y_2 + 1 = -1 \iff y_2 = -2 - y_1$

$\Rightarrow -u(-x_1, -2 - y_1)$

$-u + u = (-x_1, -2 - y_1) + (x_1, y_1) = (-x_1 + x_1, -2 - y_1 + y_1 + 1)$

$= (0, -1) \rightarrow$ Zero Vector

true Vector Space

AXIOM 6 : $k u = k(x_1, y_1) = (k x_1, y_1)$

true Vector space.

AXIOM 7 : $k(u + v) \stackrel{?}{=} k u + k v$

$k(u + v)$

$= k((x_1, y_1) + (x_2, y_2))$

$= k((x_1 + x_2, y_1 + y_2 + 1))$

$= (k x_1 + k x_2, y_1 + y_2 + 1)$

$k u + k v$

$= k(x_1, y_1) + k(x_2, y_2)$

$= (k x_1, y_1) + (k x_2, y_2)$

$= (k x_1 + k x_2, y_1 + y_2 + 1)$

true Vector space.

AXIOM 8 : $(k + m) u \stackrel{?}{=} k u + m u$

$(k + m) u$

$= (k + m)(x_1, y_1)$

$= ((k + m) x_1, y_1)$

$= (k x_1 + m x_1, y_1)$

$k u + m u$

$= k(x_1, y_1) + m(x_1, y_1)$

$= (k x_1, y_1) + (m x_1, y_1)$

$= (k x_1 + m x_1, 2 y_1 + 1)$

$\neq$

False

It fails Not vector space

AXIOM 9 : $k(m u) = (k m) u$

$k(m(x_1, y_1))$

$= k(m x_1, y_1)$

$= (k m)(x_1, y_1)$

$= (k m x_1, y_1)$

true Vector Space

AXIOM 10 : $1 \cdot u = u$

$$1 \cdot (x_1, y_1) = (x_1, y_1) \quad \Big| \quad (x_1, y_1)$$

true ✓

Vector Space ?

Is it or not a vector space?

$$\left(\begin{array}{ccc|cc} 2 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ -2 & -1 & -1 & -2 & -1 \end{array}\right)$$

$$2 + x + (-1) \quad -2 - (-2)$$