$\boxed{120}$

Exam #1 Grade 86

~~156a~~ 215 MAT

1) Solve the following system of equations by Gauss-Jordan elimination:

$$\begin{aligned} x_1 + x_2 - 2x_3 - 2x_4 - 2x_5 &= 2 \\ 3x_1 + 2x_2 - 3x_3 - 2x_4 - 8x_5 &= 1 \\ 2x_1 - x_2 + 5x_3 - x_4 - x_5 &= 16 \\ x_1 + x_3 - 4x_4 + 2x_5 &= 15 \end{aligned}$$ **(25 points)**

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the system corresponds to the matrix;

$$\begin{pmatrix} 1 & 1 & -2 & -2 & -2 & 2 \\ 3 & 2 & -3 & -2 & -8 & 1 \\ 2 & -1 & 5 & -1 & -1 & 16 \\ 1 & 0 & 1 & -4 & 2 & 15 \end{pmatrix} \xrightarrow[-r_3 \text{ to } r_4]{-3r_1 \text{ to } r_2} \begin{pmatrix} 1 & 1 & -2 & -2 & -2 & 2 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 2 & -1 & 5 & -1 & -1 & 16 \\ -1 & 1 & -4 & -3 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow[r_1 \text{ to } r_4]{-2r_1 \text{ to } r_3} \begin{pmatrix} 1 & 1 & -2 & -2 & -2 & 2 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 0 & -3 & 9 & 3 & 3 & 12 \\ 0 & 2 & -6 & -5 & 1 & 1 \end{pmatrix} \xrightarrow[-3r_2 \text{ to } r_3]{r_2 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 1 & 2 & -4 & -3 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 0 & 0 & 0 & -9 & 9 & 27 \\ 0 & 2 & -6 & -5 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{9} r_3]{2r_2 \text{ to } r_4} \begin{pmatrix} 1 & 0 & 1 & 2 & -4 & -3 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 0 & 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 3 & -3 & -9 \end{pmatrix} \xrightarrow{\frac{1}{3} r_4} \begin{pmatrix} 1 & 0 & 1 & 2 & -4 & -3 \\ 0 & -1 & 3 & 4 & -2 & -5 \\ 0 & 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[\substack{4r_3 \text{ to } r_2 \\ 2r_3 \text{ to } r_1}]{r_3 \text{ to } r_4} \begin{pmatrix} 1 & 0 & 1 & 0 & -2 & 3 \\ 0 & -1 & 3 & 0 & 2 & 7 \\ 0 & 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

let $x_5 = a$ and $x_3 = b$

then $x_1 = 3 + 2a - b$

$x_2 = -7 + 3b + 2a$

$x_4 = -3 + a$

then the solution is given by:

$$\begin{pmatrix} 3 + 2a - b \\ -7 + 3b + 2a \\ b \\ -3 + a \\ a \end{pmatrix}$$

1) a) Show that if $Ax = b$ has more than one solution, then it has infinitely many solutions. **(7 points)**

b) Give an example of 2×2 matrices A and B such that $AB = 0$, but $A \neq 0$ and $B \neq 0$. **(7 points)**

c) Show that if A is skew-symmetric, then A^n is symmetric if n is even, and skew symmetric if n is odd. **(7 points)**

a) $Ax = b$.

let x_1 and x_2 be 2 solutions of $Ax = b$.

$Ax_1 = b$. $\quad Ax_2 = b$.

~~then $Ax_1 = Ax_2$~~

b) $A \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \quad B \begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix}$ A and B are 2x2 matrices with $A \neq 0$ and $B \neq 0$.

~~$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} = \begin{matrix} 0 & 0 \\ 0 & \end{matrix}$~~

$AB \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.

c) A is skew-symmetric $\Rightarrow A^t = -A$.

* $(A^n)^t = (A^t)^n = (-A)^n$.
* if n is even then: $(-A)^n = A^n$ then $(A^n)^t = A^n$ and A^n is symmetric
* if n is odd then: $(-A)^n = -A^n$ then $(A^n)^t = -A^n$ and A^n is skew-symmetric.

3) Use the inverse method, i.e. use A^{-1}, to solve the system

$$\begin{aligned} x+2y+3z&=3\\ 2x+5y+7z&=6\,. \qquad \textbf{(20 points)}\\ 3x+7y+8z&=5 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3\\ 2 & 5 & 7 & 6\\ 3 & 7 & 8 & 5 \end{array}\right) = (A|b) \quad \text{but } Ax = b.$$

~~As~~ and ~~[illegible]~~ since A has an inverse A^{-1}, then:

$x = A^{-1}b.$

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$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3\\ 2 & 5 & 7\\ 3 & 7 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 3\\ 6\\ 5 \end{pmatrix}$$

$$* \begin{pmatrix} 1 & 2 & 3\\ 2 & 5 & 7\\ 3 & 7 & 8 \end{pmatrix} \xrightarrow[-3r_1 \text{ to } r_3]{-2r_1 \text{ to } r_2} \begin{pmatrix} 1 & 2 & 3\\ 0 & 1 & 1\\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[-r_2 \text{ to } r_3]{-2r_2 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}r_3} \begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[-r_3 \text{ to } r_1]{-r_3 \text{ to } r_2} \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

$$I_3 \xrightarrow[-3r_1 \text{ to } r_3]{-2r_1 \text{ to } r_2} \begin{pmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1 \end{pmatrix} \xrightarrow[-r_2 \text{ to } r_3]{-2r_2 \text{ to } r_1} \begin{pmatrix} 5 & -2 & 0\\ -2 & 1 & 0\\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{2}r_3} \begin{pmatrix} 5 & -2 & 0\\ -2 & 1 & 0\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\xrightarrow[-r_3 \text{ to } r_1]{-r_3 \text{ to } r_2} \begin{pmatrix} \frac{9}{2} & -\frac{5}{2} & \frac{1}{2}\\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = A^{-1}.$$

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & -\frac{5}{2} & \frac{1}{2}\\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 3\\ 6\\ 5 \end{pmatrix} = \begin{pmatrix} 1\\ -2\\ 2 \end{pmatrix}$$

4) a) Let A be an $n \times n$ matrix satisfying $k_0 I + k_1 A + k_2 A^2 + \ldots + k_m A^m = 0$, such that $k_0 \neq 0$. Show that A is invertible, and find the inverse A^{-1} of A in terms of A. **(7 points)**

b) Show that if $\bar{x},\ \bar{x}',\ \bar{x}''$ are solution for the system $A\bar{x} = 0$, then for any three numbers k_1, k_2, k_3 we have that $k_1\bar{x} + k_2\bar{x}' + k_3\bar{x}''$ is another solution for the system. **(7 points)**

a) $k_0 I + k_1 A + k_2 A^2 + \ldots + k_m A^m = 0$. $\quad k_0 \neq 0$.

for $m = 1$. $k_0 I + k_1 A = 0$.

assuming A^{-1} is the inverse of A, then: $k_0 (A^{-1}A) + k_1 A = 0$

$k_0 A A^{-1} + k_1 A = 0$

$A(k_0 A^{-1} + k_1 I) = 0$ ✗

* $(k_0 I + k_1 A + k_2 A^2 + \ldots + k_m A^m)^{-1} = 0$

$$\frac{(A^{-1})^m}{k_m} + \ldots + \cancel{\frac{(A^{-1})^2}{k_2}} + \cancel{\frac{(A^{-1})}{k_1}} + \frac{I}{k_0} = 0.$$

b) $A\vec{x} = 0$. with solutions $\vec{x}, \vec{x}', \vec{x}''$.

$A\vec{x}' = 0$

$A\vec{x}'' = 0$.

$k_1 A\vec{x} = 0 \Rightarrow A \cancel{k_1 \vec{x} = 0}$ (1)

$k_2 A\vec{x}' = 0 \Rightarrow A k_2 \vec{x}' = 0$ (2)

$k_3 A\vec{x}'' = 0 \Rightarrow A k_3 \vec{x}'' = 0$ (3)

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$\Rightarrow$

by additionning equations (1), (2), (3), we have.

$$A k_1 \vec{x} + A k_2 \vec{x}' + A k_3 \vec{x}'' = 0.$$

$$A \left(k_1 \vec{x} + k_2 \vec{x}' + k_3 \vec{x}'' \right) = 0.$$

then $k_1 \vec{x} + k_2 \vec{x}' + k_3 \vec{x}''$ is another solution for the system

5) Determine the values of k for which the following system has no solution, one solution, or infinitely many solutions

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$$\begin{aligned} -2x-kz&=1\\ x+y+z&=k \quad \text{(20 points)}\\ x-y+3z&=2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} -2 & 0 & -k & 1\\ 1 & 1 & 1 & k\\ 1 & -1 & 3 & 2 \end{array}\right) \xrightarrow{-\frac{1}{2}r_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{k}{2} & -\frac{1}{2}\\ 1 & 1 & 1 & k\\ 1 & -1 & 3 & 2 \end{array}\right)$$

$$\xrightarrow[-r_1 \text{ to } r_3]{-r_1 \text{ to } r_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{k}{2} & -\frac{1}{2}\\ 0 & 1 & (1-\frac{k}{2}) & (\frac{1}{2}+k)\\ 0 & -1 & (3-\frac{k}{2}) & \frac{5}{2} \end{array}\right) \xrightarrow{r_2 \text{ to } r_3} \left(\begin{array}{ccc|c} 1 & 0 & \frac{k}{2} & -\frac{1}{2}\\ 0 & 1 & (1-\frac{k}{2}) & (\frac{1}{2}+k)\\ 0 & 0 & (4-k) & (3+k) \end{array}\right)$$

~~$r(A) = 2$~~.

for $4-k=0$:
$k=4$. then

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -\frac{1}{2}\\ 0 & 1 & -1 & \frac{9}{2}\\ 0 & 0 & 0 & 7 \end{array}\right) \to \text{impossible} \quad \xrightarrow{\frac{1}{7}r_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & -\frac{1}{2}\\ 0 & 1 & -1 & \frac{9}{2}\\ 0 & 0 & 0 & 1 \end{array}\right) \quad r(A)=2 < r(A|b)$$

there is no solution.

for $3+k=0$:
$k=-3$.
$(4-k \neq 0)$

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & -\frac{1}{2}\\ 0 & 1 & \frac{5}{2} & \frac{7}{2}\\ 0 & 0 & 7 & 0 \end{array}\right) \xrightarrow{\frac{1}{7}r_3} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & -\frac{1}{2}\\ 0 & 1 & \frac{5}{2} & \frac{7}{2}\\ 0 & 0 & 1 & 0 \end{array}\right)$$

$r(A) = 3 = r(A|b) =$ number of variable then the system has one and only solution

$\longrightarrow$

$\rightarrow$ * ~~for $3+k\neq 0$:~~

$(4-k)\neq 0$)

$$\cancel{\left(\begin{array}{ccc|c} 1 & 0 & \frac{k}{2} & -\frac{1}{2} \\ 0 & 1 & \left(\frac{4-k}{2}\right) & \left(\frac{1}{2}+k\right) \\ 0 & 0 & (4-k) & (3+k) \end{array}\right) \xrightarrow{\left(\frac{1}{3+k}\right) r_3} \left(\begin{array}{ccc|c} 1 & 0 & \frac{k}{2} & -\frac{1}{2} \\ 0 & 1 & \left(\frac{4-k}{2}\right) & \left(\frac{1}{2}+k\right) \\ 0 & 0 & \left(\frac{4-k}{3+k}\right) & 1 \end{array}\right)}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{k}{2} & -\frac{1}{2} \\ 0 & 1 & \left(\frac{4-k}{2}\right) & \left(\frac{1}{2}+k\right) \\ 0 & 0 & (4-k) & (3+k) \end{array}\right)$$

$r(A) = 2 = r(A|b) = 2 <$ number of variables (3).

$2 < 3$.

Then for $3+k\neq 0$

$k\neq -3$, we have infinitely many solutions.