

**NDU**

**MAT 215**  
**Linear Algebra I**

**Exam # 1**

**Wednesday November 16, 2004**

**Duration: 75 minutes**

**Name:** Amanda Abi Nader

**Section:** C

**Instructor:** Kreidys Michel

**Grade:** 98

Problem Number	Points	Score
1	16	16
2	15	15
3	15	15
4	14	14
5	6	6
6	16	14
7	18	18
Total	100	

98

1) (16 points) Use Gauss-Jordan elimination to solve the following system of equations:

$$x_1 + x_2 + 3x_3 - x_4 = 0$$

$$x_1 - x_2 - x_3 - x_4 - 2x_5 = 4$$

$$x_2 + 2x_3 + 2x_4 - x_5 = 0$$

$$2x_1 - x_2 + x_4 - 6x_5 = 9$$

(1b)

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 1 & -1 & -1 & -1 & -2 \\ 0 & 1 & 2 & 2 & -1 \\ 2 & -1 & 0 & 1 & -6 \\ \end{array} \right)$$

✓

$$\begin{aligned} R_1 - R_2 &\rightarrow R_2 \\ -2R_1 + R_4 &\rightarrow R_4 \end{aligned}$$

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 0 & 2 & 4 & 0 & -4 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & -3 & -6 & 3 & -9 \\ \end{array} \right)$$

✓

$$\begin{aligned} \frac{R_2}{2} &\rightarrow R_2 \\ -\frac{R_4}{3} &\rightarrow R_4 \end{aligned}$$

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & -1 & -3 \\ \end{array} \right)$$

✓

$$R_3 - R_4 \rightarrow R_4$$

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 3 & -3 \\ \end{array} \right)$$

✓

$$R_2 - R_3 \rightarrow R_3$$

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 3 & -3 \\ \end{array} \right)$$

✓

$$\begin{aligned} -R_3/2 &\rightarrow R_3 \\ R_4/3 &\rightarrow R_4 \end{aligned}$$

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ \end{array} \right)$$

✓

$$R_3 - R_4 \rightarrow R_4$$

$$\left( \begin{array}{ccccc} 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \end{array} \right)$$

✓

$$R_1 - R_2 \rightarrow R_1$$

$$\left( \begin{array}{ccccc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \end{array} \right)$$

✓

$$R_1 + R_3 \rightarrow R_1$$

$$\left( \begin{array}{ccccc} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \end{array} \right)$$

✓

$$\begin{aligned} x_1 + x_3 - 2x_5 &= 3 \\ x_1 &= 3 - x_3 + 2x_5 \end{aligned}$$

$$\text{let } x_5 = t$$

$$x_4 - x_5 = 1$$

$$x_4 = t + 1$$

$$\text{let } x_3 = s$$

$$x_2 + 2x_3 + x_5 = -2$$

$$x_2 = -2 - 2s - 5t$$

- 2) (15 points) Discuss, according to the values of  $k$ , the nature of the solutions of the system whose augmented matrix is below. Then, solve the system whenever possible.

15

$$\left[ \begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & k^2 & k \end{array} \right]$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & k^2 & k \end{array} \right)$$

$$-2R_1 + R_3 \rightarrow R_3$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & k^2 - 2k & k \end{array} \right)$$



$$R_2 + R_3 \rightarrow R_3$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & k^2 - 4k & k \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & k(k-4) & k \end{array} \right)$$



$\boxed{k=4}$   $\Rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 8 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{array} \right)$  the system has no solution.

$\boxed{k \neq 4}$   $\Rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$  there is a free variable the system has an infinite number of solutions

Let  $x_3 = t$

$$\begin{aligned} x_2 &= 2 \\ x_1 &= 1 - 2t \end{aligned}$$



$\boxed{k \neq 4 \text{ and } k \neq 0}$   $\left\{ \begin{array}{l} x_3 = t \\ x_2 = 2 \\ x_1 = 1 - 2t \end{array} \right. \quad \left. \begin{array}{l} 1 & 0 & 2t & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & k(k-4) & k \end{array} \right\}$  one solution

$$x_3 = \frac{1}{k-4}$$

$$\begin{aligned} x_2 &= 2 \\ x_1 + \frac{2k}{k-4} &= 1 \end{aligned}$$

$$x_1 = 1 - \frac{2k}{k-4}$$



3) (15 points) Consider the matrix  $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ .

$$I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) (7 points) Find elementary matrices  $E_1$  and  $E_2$  such that  $E_2 E_1 A = I$ .

$$\textcircled{15} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

$$-3R_3 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

we do it

b) (8 points) Use part (a) to express the matrix  $A$  as a product of two elementary matrices.

$$A = E_1^{-1} E_2^{-1} I$$

$$E_1^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

$$E_2^{-1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

$\equiv$

4) (14 points)

a) (7 points) Let  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}$ . Find  $A^{-1}$  or prove that  $A$  is not invertible.

$$A \left( \begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \\ 1 & -2 & 3 & 0 & 0 \end{array} \right)$$

$$R_1 - R_3 \rightarrow R_3$$

$$-2R_1 + R_2 \rightarrow R_2$$

$$\left( \begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & -4 & 0 & 0 & -1 \end{array} \right)$$

$$R_2 - R_3 \rightarrow R_3$$

$$\left( \begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 3 & 1 & 1 \end{array} \right)$$

(7)

$\Downarrow$   
This cannot be an identity matrix  
so  $A$  is not invertible

b) (7 points) Find the matrix  $X$  such that:

$$\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Let  $A$  be  $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} -5 & 2 & 10 \\ 3 & -1 & 0 \end{pmatrix} \quad R_1 \times (-1/5) \rightarrow R_1$$

$$A = \begin{pmatrix} 1 & -2/5 & -1/5 \\ 3 & -1 & 0 \end{pmatrix} \quad R_2 + 3R_1 \rightarrow R_2$$

$$A = \begin{pmatrix} 1 & -2/5 & -1/5 & 6 \\ 0 & 1/5 & 3/5 & 1 \end{pmatrix} \quad R_2 \times 5 \rightarrow R_2$$

$$A = \begin{pmatrix} 1 & -2/5 & -1/5 & 6 \\ 0 & 1 & 3 & 5 \end{pmatrix} \quad 2/5 R_2 + R_1 \rightarrow R_1$$

$$A^{-1} = \begin{pmatrix} 1/2 \\ 3/5 \end{pmatrix}$$

$$X = A^{-1} \begin{pmatrix} 2 & 0 & -1 \\ 3 & -5 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 1/2 \\ 3/5 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 3 & -5 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -10 & -1 \\ 21 & -25 & -3 \end{pmatrix}$$

- 5) (6 points) Prove the following theorem: If  $A\vec{x} = \vec{b}$  is a system of linear equations, then the system has either no solutions, exactly one solution, or infinitely many solutions.

①  $A\vec{x} = \vec{b}$

let  $x_1$  and  $x_2$  be 2 solution.

$$A\vec{x}_1 = \vec{b}$$

$$A\vec{x}_2 = \vec{b}$$

$$A\vec{x}_1 - A\vec{x}_2 = \vec{0}$$

$$A(\vec{x}_1 - \vec{x}_2) = \vec{0}$$

$\vec{x}_1 - \vec{x}_2$  is a solution of  $A\vec{x} = \vec{0}$

so  $\vec{x}_1 - \vec{x}_2$  is a solution of  $A\vec{x} = \vec{0}$  so the system has exactly one solution.

②  $A\vec{x} = \vec{b}$

$x_1$  and  $x_2$  are 2 solution.

$$A(\vec{x}_1 - \vec{x}_2) = \vec{0} \quad \text{let } \vec{x}_0 = \vec{x}_1 - \vec{x}_2$$

$$A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + kA\vec{x}_0 = \vec{b} \quad \text{so } (\vec{x}_1 + k\vec{x}_0) \text{ is a solution of } A\vec{x} = \vec{b}$$

$k$  can take any value so the system has infinitely many solutions.

if  $\vec{x}_1 = \vec{x}_2$

if  $\vec{x}_1 \neq \vec{x}_2$  ③  $A\vec{x} = \vec{b}$

then there is no solution

6) (16 points) Let  $A$  be an  $n \times n$  matrix. Prove each of the following statements:

a) (4 points) If  $A$  is a diagonal matrix all of whose entries are  $d$ , then  $\text{tr}(A) = nd$ .

$$A \begin{pmatrix} d & & & \\ d & d & & \\ d & d & d & \\ \vdots & & & \ddots \\ A & & & \end{pmatrix}$$

$$B \begin{pmatrix} d & & & & \\ d & d & & & \\ d & d & d & & \\ d & d & d & d & \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ d & d & d & d & \end{pmatrix}$$

$$\text{tr}(A) = \sum a_{ii}$$

$$= \sum d \cdot \sum a_{ii}$$

If there is  $m$  entries then

$$\text{tr}(A) = d + d + d + d + \dots$$

$$\text{So } \boxed{\text{tr}(A) = md}$$

b) (4 points)  $A - A^T$  is skew-symmetric.

(4)

$$(A - A^T)^T = A^T - (A^T)^T$$

$$= A^T - A$$

$$= -(A - A^T)$$

so then it's skewsymmetric.

c) (4 points) If  $A$  is invertible and  $k \neq 0$  is a scalar, then  $kA^{-n}$  is invertible and

$$(kA^{-n})^{-1} = \frac{1}{k} A^n.$$

$$(KA^{-n})^{-1} = K^{-1} A^n = (KA^{-n})^{-1} \frac{1}{K} K = \frac{1}{K} (K^{-1} A^n) = (A^{-n})^{-1} = A^n$$

d) (4 points) Let  $I$  be the  $n \times n$  identity matrix. Then  $IA = A$ .

(2)

$$A \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AI = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

particular case

$$\text{So } \boxed{IA = A}$$

$$A \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A^T \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\xrightarrow{\Sigma A = A}$$

$$\begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & \vdots & \ddots & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \times & \times \end{pmatrix} = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \vdots & \vdots & \ddots \\ \times & \times & \times \end{pmatrix}$$

7) (18 points) Let  $A$  and  $B$  be two  $n \times n$  matrices. Prove each of the following statements.

a) (4 points) If  $A$  and  $B$  are both invertible, then  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$ .

(4)

$(AB)^{-1} = B^{-1}A^{-1}$  because:  
The product of invertible matrix is invertible.

$$\boxed{AA^{-1} = I}$$

$$\boxed{BB^{-1} = I}$$

$$(AB)(AB)^{-1} = I \text{ so } AB \text{ is invertible.}$$

$$\text{and } (AB)^{-1} = B^{-1}A^{-1}$$

b) (4 points) If  $A$  is invertible and  $B$  is row-equivalent to  $A$ , then  $B$  is invertible.

(4)

$B$  is row equivalent to  $A$ .

$$\text{so } B \sim A = E_1 \dots E_n E_{n-1} \dots E_1 B.$$

So the product of elementary matrix  
matrices and an invertible matrix

$$B = E_1^{-1} E_2^{-1} E_3^{-1} \dots E_n^{-1} A.$$

the product of invertible elementary matrices all invertible  
is invertible  $\Rightarrow B$  is invertible

c) (6 points) Let  $A\vec{x} = \vec{0}$  be a homogeneous system of linear equations and let  $B$  be invertible. Show that if  $A\vec{x} = \vec{0}$  has only the trivial solution, then  $(BA)\vec{x} = \vec{0}$  has only the trivial solution.

(6)

If  $A\vec{x} = \vec{0}$  has only the trivial solution then  
 $A$  is invertible.

Then  $BA$  is invertible because  $A$  and  $B$  are invertible

so  $(BA)\vec{x} = \vec{0}$  has only the  
trivial solution because

if  $(BA)$  is invertible then

$$(BA)^{-1}(BA)\vec{x} = \vec{0} \Rightarrow A^{-1}B^{-1}BA\vec{x} = \vec{0}$$

$$\Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0} \Rightarrow \text{trivial solution}$$

d) (4 points) A square matrix  $M$  is said to be orthogonal if  $MM^T = I$ . Show that if  $A$  and  $B$  are both orthogonal, then  $AB$  is also orthogonal.

$A$  is orthogonal so  $AA^T = I$

$B$  is orthogonal so  $BB^T = I$

(4)

$$\begin{aligned} (AB)(AB)^T &= AB(B^T A^T) \\ &= AA^T BB^T \\ &= I \end{aligned}$$

then  $AB$  is also orthogonal.