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Department of Mathematics and Statistics
Math 215: Linear Algebra I
Exam 1: 19 November, 2001
Time: 60 minutes

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Please write your answer in the space below the question and on the back of the sheet, if you need more space. Use the attached white sheets only for scratch work. Problems 1, 2, and 3 have 30 points each, and problem 4 has 10 points.

1) Solve the following linear system by Gauss-Jordan elimination:

$$x + y + z = 1$$
$$x + y - z = 3$$
$$3x + z = 2$$
$$x - 2y - z = 0$$

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ 3 & 0 & 1 & 2 \\ 1 & -2 & -1 & 0 \end{pmatrix} \xrightarrow[\substack{-r_1 \text{ to } r_4 \\ -3r_1 \text{ to } r_3}]{-r_1 \text{ to } r_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -3 & -2 & -1 \end{pmatrix} \xrightarrow{-r_3 \text{ to } r_4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2 \text{ to } r_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[-\frac{1}{3} r_3]{-\frac{1}{2} r_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-r_3 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-r_2 \text{ to } r_1} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{interchange } r_2 \text{ and } r_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$\Rightarrow z = -1 \; ; \; y = 1 \; , \; x = 1$

$S : (x, y, z) = (1, 1, -1)$

2) a) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$.

b) Find elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = I$.

c) Write A as a product of elementary matrices (find these matrices).

a)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{-r_1 \text{ to } r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2r_3 \text{ to } r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-r_3 \text{ to } r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-r_1 \text{ to } r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{-2r_3 \text{ to } r_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{-r_3 \text{ to } r_1} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} =$$

b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-r_1 \text{ to } r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2r_3 \text{ to } r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-r_3 \text{ to } r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

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* $I \xrightarrow{e_1} E_1$

$A \xrightarrow{e_1} B$

$\Rightarrow E_1A = B$

* $I \xrightarrow{e_2} E_2$

$E_1A \xrightarrow{e_2} C$

$\Rightarrow E_2E_1A = C$.

* $I \xrightarrow{e_3} E_3$

$E_2E_1A \xrightarrow{e_3} I_3$

$\Rightarrow E_3E_2E_1A = I$.

c) $E_3E_2E_1A = I \Rightarrow E_1^{-1}E_2^{-1}\underbrace{E_3^{-1}E_3}_{I}E_2E_1A = E_1^{-1}E_2^{-1}E_3^{-1}I$.

$\Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

E_1^{-1}: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \text{ to } r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = E_1^{-1}$.

E_2^{-1}: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2r_3 \text{ to } r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ E_2^{-1}.

E_3^{-1}: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 \text{ to } r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3^{-1}$.

3) a) Show that if $ad - bc \neq 0$ then the reduced row echelon form of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is I_2.

b) Show that if A is $n \times n$ with a zero row, then A is not invertible.

c) Show that if A is $n \times n$ and $A^2 - 3A - I = 0$ then A is invertible. Find A^{-1}.

d) Show that if A is $n \times n$ such that $A^T A = A$ then A is symmetric, and $A^2 = A$.

a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse : $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

this exist $ad - bc \neq 0$

$ad - bc \neq 0 \Rightarrow$ the inverse existe $\Rightarrow$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$.

we make some elementary row operation on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ (the same e.r.o on I_2 gives the inverse).

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b) A is an $n \times n$ matrix with a zero row

$\Rightarrow$ in rref this zero row still exist

then $r(\text{rref}(A)) < n$

$\Rightarrow A$ is not invertible

c) $A^2 - 3A - I = 0 \Rightarrow A^2 - 3A = I \Rightarrow A$ is invertible.

cause if A has a ~~non~~ zero row then A^2 has a ~~non~~ zero row + as well as $(-3A) \Rightarrow I$ has an zero row which is not tr

$\Rightarrow A$ does not have a zero row $\Rightarrow r(\text{rref}(A)) = n \Rightarrow A$ is in

- $A^2 - 3A = I \Rightarrow A^{-1} I = A^{-1}A^2 - 3A^{-1}A = A - 3I = A^{-1}$

$$\boxed{A^{-1} = A - 3I}$$

d) $A^T A = A \Rightarrow (A^T A)^T = A^T$

but : $(A^T A)^T = (A^T) \cdot (A^T)^T = A^T \cdot A \neq A^T$

4) Let $A=\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} -1 & -1 \\ 5 & 6 \\ 1 & 0 \end{pmatrix}$. Evaluate

a) $(6A+2B^T)^T$ b) $tr\left(\frac{1}{3}(AB)^{-1}\right)$

a) $6A = \begin{vmatrix} 12 & 6 & 18 \\ -6 & 0 & 6 \end{vmatrix}$

$B^T = \begin{pmatrix} -1 & 5 & 1 \\ -1 & 6 & 0 \end{pmatrix} \Rightarrow 2B^T = \begin{pmatrix} -2 & 10 & 2 \\ -2 & 12 & 0 \end{pmatrix}$

$6A + 2B^T = \begin{pmatrix} 12 & 6 & 18 \\ -6 & 0 & 6 \end{pmatrix} + \begin{pmatrix} -2 & 10 & 2 \\ -2 & 12 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 16 & 20 \\ -8 & 12 & 6 \end{pmatrix}$

$(6A+2B^T)^T = \begin{pmatrix} 10 & -8 \\ 16 & 12 \\ 20 & 6 \end{pmatrix}$ ✓

b) $AB = \underset{2\times3}{\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}} \underset{3\times2}{\begin{vmatrix} -1 & -1 \\ 5 & 6 \\ 1 & 0 \end{vmatrix}} = \underset{2\times2}{\begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}}$

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$AB^{-1} = \frac{1}{6-8} = -\frac{1}{2}\begin{pmatrix} 1 & -4 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} -1/2 & 2 \\ 1 & -3 \end{pmatrix}$

$\frac{1}{3}(AB)^{-1} = \frac{1}{3}\begin{pmatrix} -1/2 & 2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -1/6 & 2/3 \\ \frac{1}{3} & -1 \end{pmatrix}$

$tr\left(\frac{1}{3}(AB)^{-1}\right) = -\frac{1}{6} - 1 = \boxed{-\frac{7}{6}}$ ✓