1) (16 points) Use Gauss-Jordan elimination to solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + 3x_3 - x_4 \quad &= 0 \\ x_1 - x_2 - x_3 - x_4 - 2x_5 &= 4 \\ x_2 + 2x_3 + 2x_4 - x_5 &= 0 \\ 2x_1 - x_2 \quad + x_4 - 6x_5 &= 9 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ 1 & -1 & -3 & -1 & -2 \\ & & & & \\ & & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 & 0 \\ 1 & -1 & -1 & -1 & -2 & 4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 & -6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & -2 & -4 & 0 & -2 & 4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -3 & -6 & 3 & -6 & 9 \end{bmatrix} \times -1/2$$

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -3 & -6 & 3 & -6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 3 & -3 & 3 \end{bmatrix} \begin{matrix} \times 1/2 \\ \times 1/3 \end{matrix}$$

no need

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free var

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -2 & 3 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

let x_5, x_2 free variable let $x_5 = t$, $x_3 = r$

2) (15 points) Discuss, according to the values of k, the nature of the solutions of the system whose augmented matrix is below. Then, solve the system whenever possible.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & k^2 & k \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & k^2-4k & k-2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4k+k^2 & k \end{array}\right]$$

~~$2k-4k^2 = 2k(1-2k)$~~

$k(k-4)$

* If $k=0$ Then infinitely many solution: → free variable

let $x_3 = r$, $x_2 = 2$, $x_1 = 1$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ r \end{pmatrix}$

* If $k = 4$ Then The Third row will become $0 \quad 0 \quad 0 \mid 4$ Then no solution Inconsistent system

* If $k \neq 4 \neq 0$ unique solution

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$$\left[\begin{array}{ccc|c} 1 & 0 & 2k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{k}{k^2-4k} \end{array}\right]$$

~~$x_3 = \frac{1}{2-4k}$, $x_2 = 2$, $x_1 = 1-2k\,x_3$~~

$x_3 = \frac{k}{k^2-4k} = \frac{1}{k-4}$

$x_2 = 2$, $x_1 = 1 - 2k\left(\frac{1}{k-4}\right) = \frac{k-4-2k}{k-4} = \frac{-k-4}{k-4}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -k-4/k-4 \\ 2 \\ 1/k-4 \end{pmatrix}$$

3) (15 points) Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

a) **(7 points)** Find elementary matrices E_1 and E_2 such that $E_2 E_1 A = I$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & & 0 \\ & & \\ & & \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad E_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) **(8 points)** Use part (a) to express the matrix A as a product of two elementary matrices.

E_2 and E_1 invertible Then $E_2 E_1$ also invertible

$E_2 E_1 A = I$ Then $(E_2 E_1)^{-1} E_2 E_1 A = (E_2 E_1)^{-1} I$

$A = E_1^{-1} E_2^{-1} I = E_1 E_2^{-1}$

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

15

4) (14 points)

a) **(7 points)** Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}$. Find A^{-1} or prove that A is not invertible.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -1 & +2 & -1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right]$$

So A is not invertible since it can not be reduced to I.

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b) **(7 points)** Find the matrix X such that: $\overset{A}{\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}} X = \overset{B}{\begin{bmatrix} 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}}$.

$$A = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \qquad A^{-1} = \frac{1}{5-6}\begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$AX = B \qquad \underbrace{A^{-1}A}_{I}X = A^{-1}B \longrightarrow X = A^{-1}B$$

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$$X = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -10 & -1 \\ 21 & -25 & -3 \end{bmatrix}$$

14

5) **(6 points)** Prove the following theorem: If $A\vec{x} = \vec{b}$ is a system of linear equations, then the system has either no solutions, exactly one solution, or infinitely many solutions.

-1 * If $A\vec{x}=\vec{b}$

The problem can be solved if we shows That $A\vec{x}=\vec{b}$ has infinitely many solution

let x_1, x_2 be solutions for $A\vec{x}=\vec{b}$, with $\vec{x_1}\neq\vec{x_2}$

$-\frac{1}{2}$

Then $A\vec{x_2} - A\vec{x_1} = A(\vec{x_2}-\vec{x_1}) = \vec{b}-\vec{b} = \vec{0}$

So $\vec{x_2}-\vec{x_1}$ is a solution of $A\vec{x}=\vec{0}$.

let x_0 This solution and K a scalar

$4\frac{1}{2}$

$A(\vec{x_1}+K\vec{x_0}) = A\vec{x_1} + KA\vec{x_0} = \vec{b} + K\times 0 = \vec{b}$

Then $\vec{x_1}+K\vec{x_0}$ is a solution for The sys. $A\vec{x}=\vec{b}$

or K may have many ~~mantille~~ values

Then The system has infinetly many ~~solutions~~

6) (16 points) Let A be an $n\times n$ matrix. Prove each of the following statements:

a) **(4 points)** If A is a diagonal matrix all of whose entries are d, then $tr(A)=nd$.

Diagonal matrix: The only are Known entries on The diagonal

$\mathrm{Tr}(A) = \sum A_{ii} = \underbrace{d+d+\dots+d}_{n\ \text{times}} = nd.$

The have n entries since non of them is 0 and each entry is d Then $\mathrm{Tr}(A)=$

4

b) **(4 points)** $A-A^T$ is skew-symmetric.

Skew symmetric if $A^T=-A$.

$(A-A^T)^T = A^T-(A^T)^T = A^T-A = -(A-A^T).$

Then $A-A^T$ is skew symmetric

4

c) **(4 points)** If A is invertible and $k\neq 0$ is a scalar, then kA^n is invertible and $(kA^n)^{-1}=\frac{1}{k}A^{-n}$.

~~invertible~~ ~~kA~~

If A is invertible Then A is

$(A^{-1})^n \times A^n = (A^{-1}\times A)^n = I$ } $(A^{-1})^n$ The inverse of A^n

$A^n \times (A^{-1})^n = (A\times A^{-1})^n = I$

A^n is invertible Then kA^n is invertible.

$(kA^n)\times\frac{1}{k}A^{-n} = \frac{k}{k}A^n\times(A^{-n}) = \frac{k}{k}A^0 = (I)^n = I$ and $\frac{1}{k}A^{-n}\times kA^n = \frac{k}{k}(A^{-n}A^n) = I$

The $(kA^n)^{-1} = \frac{1}{k}A^{-n}$.

4

d) **(4 points)** Let I be the $n\times n$ identity matrix. Then $IA=A$.

If A is $n\times n$ matrix.

let $B = IA$.

$b_{ij} = \sum I_{im}\times a_{mj} = \underbrace{I_{i0}\times a_{0j}}_{0} + \underbrace{I_{i1}\times a_{1j}}_{0} \dots I_{ii}\times a_{ij}$

$= I_{ii}\times a_{ij} = 1\times a_{ij} = a_{ij}$

So $B = IA$ is ~~equivalent~~ equal to A

IA is and $n\times n$ matrices

Because we have Same size $n\times n$ and same entries

4

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7) (18 points) Let A and B be two $n\times n$ matrices. Prove each of the following statements.

a) **(4 points)** If A and B are both invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

A and B invertible then A^{-1} and B^{-1} can be taken.

$AB \times B^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$ Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

4

b) **(4 points)** If A is invertible and B is row-equivalent to A, then B is invertible.

B row equivalent to A then $E_k E_{k-1} \cdots E_2 E_1 \cdot B = A$.

The product of elementary matrices is an invertible matrix. So

$(E_k E_{k-1} \cdots E_2 E_1)^{-1} (E_k E_{k-1} \cdots E_2 E_1) B = (E_k E_{k-1} \cdots E_2 E_1)^{-1} A$.

Then $B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A$; A is invertible. B the product of invertible matrices is invertible

4

c) **(6 points)** Let $A\bar{x} = \bar{0}$ be a homogeneous system of linear equations and let B be invertible. Show that if $A\bar{x} = \bar{0}$ has only the trivial solution, then $(BA)\bar{x} = \bar{0}$ has only the trivial solution.

If $A\bar{x} = \bar{0}$ has only the trivial solution then A is invertible and B is invertible. Then BA the product of 2 invertible matrices is invertible. So

Then $(BA)\bar{x} = \bar{0}$ will have only the trivial solution.

6

d) **(4 points)** A square matrix M is said to be orthogonal if $MM^T = I$. Show that if A and B are both orthogonal, then AB is also orthogonal.

A is orthogonal then $AA^T = I$

B " " " $BB^T = I$

4

$AB \times (AB)^T = \underbrace{AB \times B^T}_{\substack{I \\ \text{orthogonal}}} A^T = AIA^T = \underbrace{AA^T}_{} = I$ A is orthogonal.

Then AB is also orthogonal

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