a) (5 points) Find an equation for the level surface of this function that passes through the point (2,1,1).

$A(2,1,1)$

$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

$= \frac{x}{2} \vec{i} - 2y \vec{j} + 2z \vec{k}$

$\boxed{\vec{\nabla} f_A = \vec{i} - 2\vec{j} + 2\vec{k}}$

equation of level surface: $f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$

$(1)(x-2) + (-2)(y-1) + (2)(z-1) = 0$

$x - 2 - 2y + 2 + 2z - 2 = 0$

$\boxed{x - 2y + 2z - 2 = 0}$

$f(x,y,z) = c.$

$\frac{x^2}{4} - y^2 + z^2 = c.$

$\frac{x^2}{4c} - \frac{y^2}{c} + \frac{z^2}{c} = 1.$

b) (5 points) Identify then draw this level surface.

$x - 2y + 2z - 2 = 0$

$2z = 2y - x + 2$

$z = y - \frac{x}{2} + 1$

$D: \mathbb{R}^2$

Solution on other page.

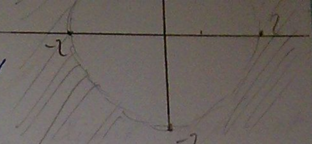
3

$\boxed{x^2+y^2 < 4}$

$x^2+y^2=4$

$\frac{x^2}{4}+\frac{y^2}{4}=1$

The domain is a circle whose center is the origin and radius = 2.



b) (2 points) Determine the boundary of the domain

2

The boundary of the domain is the disc or the circle centered at $O(0,0)$ and has $r=2$.

c) (4 points) Is the domain closed, open, or neither? Justify your answer

4

it is an open domain since the domain does not include the boundary, which is the disc.

d) (2 points) Is the domain bounded or unbounded? Justify your answer

2

The domain is unbounded since it cannot be contained in a closed disc.

d) (3 points) Find the function's range

3

for $x^2+y^2=4^+ \Rightarrow \ln(x^2+y^2-4)=-\infty$.

for $x^2+y^2=\infty \Rightarrow \ln(x^2+y^2-4)=+\infty$

So the range is $(-\infty,+\infty)$.

e) (4 points) Describe the function's level curves, and sketch a few of them.

4

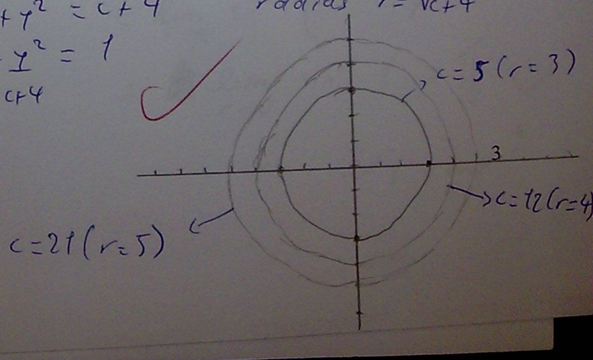
$f(x,y)=\ln(x^2+y^2-4)=c$

$x^2+y^2-4=c$

$x^2+y^2=c+4$

$\frac{x^2}{c+4}+\frac{y^2}{c+4}=1$

The level curves are circles centered at the origin with radius $r=\sqrt{c+4}$



We use the 2-path test.

$y = mx^4$

$$\Rightarrow \frac{mx^4x^4}{x^8+m^2x^8} = \frac{mx^8}{x^8(1+m^2)} = \frac{m}{1+m^2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^4y}{x^8+y^2} = \lim_{(x,y)\to(0,0)} \frac{m}{1+m^2}$$

The limit does not exist since it depends on m and will differ for different values of m.

b) (8 points)) Consider the function $f(x,y) = \sin^{-1}(1+\frac{x^4+y^4}{x^2+y^2})$

If possible, define $f(0,0)$ in a way that extends f to be continuous at the origin.

$$\cancel{\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,0)} \sin^{-1}\left(1+\frac{x^4+y^4}{x^2+y^2}\right)}$$

$$\cancel{= \lim_{(0,0)} \sin^{-1}\left(\frac{x^2+y^2+x^4+y^4}{x^2+y^2}\right)}$$

$$\cancel{= \lim_{(0,0)} \sin^{-1}\left(x^2\right)}$$

we use polar coord.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$$\cancel{\lim_{(x,y)\to(0,0)} f(x,y)}$$

$$\lim_{(0,0)} \sin^{-1}\left(1+\frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right)$$

$$= \lim_{(0,0)}\left(\sin^{-1} 1+\frac{r^4}{r^2}\right)$$

$$= \lim_{r\to 0} \sin^{-1}(1+r^2)$$

$$= \lim_{r\to 0} \sin^{-1} 1 = \frac{\pi}{2}.$$

$\frac{\pi}{3}$ cont.

5) **(14 points)** Let $w = f(x, y)$ be a function of x and y where $x = r^2 + s^2$ and $y = \frac{s}{r}$. Find each of the following and express your answer in terms of r, s, f_x, f_y, f_{xx}, f_{xy}, and f_{yy} as needed. Assume that the necessary derivatives exist and that the functions and derivatives are all continuous.

$w_s = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dr}$

a) **(6 points)** Find w_s

$= f_x \cdot \frac{dx}{dr} + f_y \cdot \frac{dy}{dr}$

$\boxed{\frac{dx}{dr} = 2r}$

$\boxed{\frac{dy}{dr} = \frac{-s}{r^2}}$

$\boxed{w_s = f_x \cdot 2s - f_y \cdot \frac{1}{r^2}}$

X

b) **(8 points)** Find w_{ss}

$w_{ss} = \frac{(dw)^2}{d^2 s} = \frac{(\partial w)^2}{\partial^2 x} \cdot \frac{(dx)^2}{dr^2} + \frac{(\partial w)^2}{\partial^2 y} \cdot \frac{(dy)^2}{d^2 y}$

$= f_{xx} \cdot x_{rr} + f_{yy} \cdot y_{rr}$

$x_r = 2r$

$x_{rr} = 2$

$y_r = \frac{-s}{r^2}$

$y_{rr} = \frac{2s}{r^3}$

$\boxed{w_{ss} = 2f_{xx} + \frac{2s}{r^3} \cdot f_{yy}}$

X

6) **(9 points)** Estimate how much the value of $f(x,y,z)=e^x \sin y - z$ will change if the point $P(x,y,z)$ moves 0.2 units from $A(0,0,2)$ straight toward $B(1,2,0)$.

$\frac{df}{ds} = \vec{\nabla} f_P \cdot \vec{u}$ where $\boxed{ds = 0.2}$

$df = \vec{\nabla} f_P \cdot \vec{u} \cdot ds.$

$A(0,0,2)$ $B(1,2,0)$

$\vec{AB} \begin{vmatrix} 1 \\ 2 \\ -2 \end{vmatrix}$ $\vec{u} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{1+4+4}}$

$= \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$

$f(x,y,z) = e^x \sin y - z$

$f_x = e^x \sin y$ replace in A

$f_y = e^x \cos y$ $\Rightarrow f_x = 0$

$f_z = -1$ $f_y = 1$ $\Rightarrow \vec{\nabla} f_{|A} = \vec{j} - \vec{k}$

$f_z = -1$

$\vec{\nabla} f_{|A} \cdot \vec{u} = (0)(\frac{1}{3}) + (1)(\frac{2}{3}) + (-1)(-\frac{2}{3})$

$= 0 + \frac{2}{3} + \frac{2}{3}$

$= \frac{4}{3}$

$df = \frac{4}{3}(0.2)$

$= \frac{4}{15}$ units or $0.2\overline{6}$ units.

9

$\ldots + e^{yz} + \ln(x^2 y) = 1 + \ln 8$ defines z implicitly as a differentiable function of x and y, then $\dfrac{\partial z}{\partial x}$ at (2, 2, 0) will have the value:

a) $\dfrac{2}{3}$

b) $\dfrac{1}{2}$

c) $-\dfrac{2}{3}$

d) $-\dfrac{1}{2}$ $\dfrac{dz}{dx}$

$f(x,y) = x\cos y + e^{xy}$ decreases most rapidly at P(1,0) in the direction of:

a) $\vec{i} + \vec{j}$

b) $\vec{i} - \vec{j}$

c) $-\vec{i} - \vec{j}$

d) $-\vec{i} + \vec{j}$

$\nabla f = (\cos y + y e^{xy})\vec{i} + (-x\sin y + x e^{xy})\vec{j}$

$= (1 + 0)\vec{i} + (0 + 1)\vec{j}$

$= \vec{i} + \vec{j}$

An equation for the plane tangent to the surface $z = 2\sqrt{y - x}$ at P(1, 2, 2) is:

a) $-x + y - z = -1$

b) $-x + y = 1$

c) $x - y = 1$

d) $x - y - z = -1$

$f(x,y) = 2\sqrt{y-x}$ $\quad (y-x)^{1/2}$ $\quad \frac{1}{2}(y-x)^{-1/2}$

$f_x = \dfrac{-1}{\sqrt{y-x}}$ $\quad f_y = \dfrac{1}{\sqrt{y-x}}$

$f_x = -1$ $\quad f_y = 1$

$-1(x-1) + 1(y-2) - (z-2) = 0$

$-x + 1 + y - 2 - z + 2 = 0$

$y - x - z = -1$

Consider the function $f(x,y)$ such that $f_x = 8 - 2x^2$ and $f_y = 6y - 7$. Then $f(x,y)$ has:

a) a local maximum at $(-2, \frac{7}{6})$ and a saddle point at $(2, \frac{7}{6})$;

b) a local minimum at $(-2, \frac{7}{6})$ and a saddle point at $(2, \frac{7}{6})$;

c) a saddle point at $(-2, \frac{7}{6})$ and a local minimum at $(2, \frac{7}{6})$;

d) a saddle point at $(-2, \frac{7}{6})$ and a local maximum at $(2, \frac{7}{6})$;

$8 - 2x^2 = 0$ $\quad 6y - 7 = 0$

$2x^2 = 8$ $\quad 6y = 7$

$x^2 = 4$ $\quad y = \frac{7}{6}$

$x = \pm 2$

$(2, \frac{7}{6})$ $\quad (-2, \frac{7}{6})$

$f_{xx} = -4x$ $\quad$ saddle $\quad$ ~~saddle~~ min.

$f_{yy} = 6$

$f_{xy} = 0.$

$f_{xx} f_{yy} - f_{xy}^2$

$= -24x$

12 points) Let x, y, z be **nonzero** numbers ($x \neq 0, y \neq 0, z \neq 0$)
Find the largest and smallest products the numbers x, y, and z can have if $x + y + z^2 = 20$

$$f(x,y,z) = xyz$$
$$g(x,y,z) = x + y + z^2 - 20$$

We use bernolli.
where $\vec{\nabla f} = \lambda \vec{\nabla g}$

$$\vec{\nabla f} = yz\vec{i} + xz\vec{j} + xy\vec{k}$$
$$\vec{\nabla g} = \vec{i} + \vec{j} + 2z\vec{k}.$$

$$yz\vec{i} + xz\vec{j} + xy\vec{k} = \lambda\vec{i} + \lambda\vec{j} + 2z\lambda\vec{k}$$

$$\begin{cases} yz = \lambda & (1) \\ xz = \lambda & (2) \\ xy = 2z\lambda & (3) \\ g(x) = 0 & (4) \end{cases}$$

We deduce from equations (1) and (2) that $x = y$.

so we replace in (3)

$$\begin{cases} x^2 = 2z\lambda & (5) \\ xz = \lambda & (6) \end{cases}$$

in (6) $\Rightarrow x = \dfrac{\lambda}{z}$

$\Rightarrow$ in (5) $\dfrac{\lambda^2}{z^2} = 2z\lambda$

$\Rightarrow \boxed{\lambda = 2z^3}$

$-y + z^2 - 20 = 0$

$+ \dfrac{\lambda}{z} + z^2 - 20 = 0$

$+ \dfrac{2z^3}{z} + z^2 - 20 = 0$

$^2 + 2z^2 + z^2 - 20 = 0$

$z^2 = 20$

$^2 = 4$

$= \pm 2.$

$x = \dfrac{\lambda}{z} = 2z^2 = 8 \Rightarrow \boxed{x = 8}$ $\boxed{y = 8}$

so the 2 pts are
$A(8, 8, 2)$ & $B(8, 8, -2)$