15

1. (15 points) Test the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points.

$f_x = 3x^2 + 6x$; $f_y = 3y^2 - 6y$

$f_x = 0 \iff x(3x+6) = 0$; $f_y = 0 \iff y(3y-6) = 0$

$x = 0$; $3x + 6 = 0$, $3x = -6$, $x = -2$; $y = 0$, $3y = 6$, $y = 2$

the critical points are: $(0,0), (0,2), (-2,0), (-2,2)$

$f_{xx} = 6x + 6$; $f_{yy} = 6y - 6$

$f_{xy} = 0$, $f_{yx} = 0$

at $(0,0)$:

$f_{xx} f_{yy} - f_{xy}^2 = (6(0)+6)(6(0)-6) - 0 = -36 < 0$

$\therefore (0,0)$ is a saddle point

at $(0,2)$:

$f_{xx} f_{yy} - f_{xy}^2 = (6)(6) - 0 = 36 > 0$

or $f_{xx}|_{(0,2)} = 6 > 0$ so $(0,2)$ is a local Minima

at $(-2,0)$:

$f_{xx} f_{yy} - f_{xy}^2 = (-6)(-6) - 0 = 36 > 0$

$f_{xx}(-2,0) = -6 < 0$ so $(-2,0)$ is a local Maxima

at $(-2,2)$

$f_{xx} f_{yy} - f_{xy}^2 = (-6)(6) - 0 = -36 < 0$

so $(-2,2)$ is a saddle point

15

2. (15 points) Find the point closest to the origin on the line of intersection of the planes $x+y+z=1$ and $-x+y+z=1$.

$f(x,y,z) = x^2+y^2+z^2$ $\quad OM^2 = x^2+y^2+z^2$ ($\sqrt{x^2+y^2+z^2}$ having the same study as $x^2+y^2+z^2$)

$g(x,y,z) = x+y+z-1=0$ ①

$h(x,y,z) = -x+y+z-1=0$ ②

Lagrange Multipliers:

$\nabla f = \lambda \nabla g + \mu \nabla h$

$(2x, 2y, 2z) = \lambda(1,1,1) + \mu(-1,1,1)$

$\begin{cases} 2x = \lambda - \mu & \therefore x = \frac{\lambda-\mu}{2} \\ 2y = \lambda + \mu & \therefore y = \frac{\lambda+\mu}{2} \\ 2z = \lambda + \mu & \therefore z = \frac{\lambda+\mu}{2} \end{cases}$

in ①:

$\frac{\lambda-\mu}{2} + \frac{\lambda+\mu}{2} + \frac{\lambda+\mu}{2} = 1$

$\therefore \frac{\lambda}{2} - \frac{\mu}{2} + \lambda + \mu = 1 \therefore \frac{3}{2}\lambda + \frac{\mu}{2} = 1 \therefore \boxed{\mu = 2 - 3\lambda}$

in ②: $-\frac{\lambda}{2} + \frac{\mu}{2} + \lambda + \mu = 1 \therefore \frac{\lambda}{2} + \frac{3\mu}{2} = 1$

$\therefore \frac{\lambda}{2} + \frac{3}{2}(2-3\lambda) = 1$

$\frac{\lambda}{2} + 3 - \frac{9}{2}\lambda = 1$

$-\frac{8}{2}\lambda = -2$

$-4\lambda = -2 \therefore \boxed{\lambda = \frac{1}{2}}$

$\therefore \mu = 2 - 3\lambda$
$= 2 - \frac{3}{2} = \boxed{\frac{1}{2}}$

$x = \frac{\frac{1}{2}-\frac{1}{2}}{2} = 0$

$y = \frac{1}{2}$

$z = \frac{1}{2}$.

the point closest to the origin is $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

10

3. (10 points) The derivative of $f(x,y)$ at the point $P_0(1,2)$ in the direction of $i+j$ is $2\sqrt{2}$ and in the direction of $-2j$ is -3. What is the derivative of f in the direction of $-i-2j$?

$u = i + j$ ∴ unit vector $= \frac{i+j}{\sqrt{2}}$

$v = -2j$ unit vector $= \frac{-2j}{\sqrt{4}} = -j$

$D_u f|_{P_0} = 2\sqrt{2}$; $D_v f|_{P_0} = -3$.

$(f_x, f_y)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$; $(f_x, f_y)\cdot(0, -1) = -3$

$\boxed{\frac{f_x}{\sqrt{2}} + \frac{f_y}{\sqrt{2}} = 2\sqrt{2}}$; $-f_y = -3$ ∴ $\boxed{f_y = 3}$

$\frac{f_x}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 2\sqrt{2}$

$\frac{f_x}{\sqrt{2}} = 2\sqrt{2} - \frac{3}{\sqrt{2}}$

$f_x = 4 - 3 = 1$

so $D_w f|_{P_0}$ in the direct° of $w = \frac{-i-2j}{\sqrt{5}}$

$= (f_x, f_y)\cdot\left(-\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$

$= \frac{-1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = \frac{-7}{\sqrt{5}}$

3

4. (8 points) Find the points on the surface $(y+z)^2 + (z-x)^2 = 16$ where the normal line is parallel to the yz-plane. (Hint: Use Gradient).

Normal line // To the yz plane ∴ normal line is ⊥ To x.y plane

so the gradient which is ⊥ To the level curve is ⊥ To this normal line (here)

$f(x,y,z) = (y+z)^2 + (z-x)^2 - 16 = 0$

$f = (f_x, f_y, f_z) = (-2(z-x), 2(y+z), 2(y+z) + 2(z-x))$

$\nabla f = (2x, 2y, 2y-2x)$ $f_x = 0$

$-2z + 2x = 0$

Vector of N $\cdot \nabla f = 0$ $\boxed{z = x}$ all the pts on this

level curve $y^2 + x^2 = 16$ $(f_x, f_y, f_z)\cdot(1, 0, 0) = 0$ x axis

$z = x$

$y + z = \pm 4$

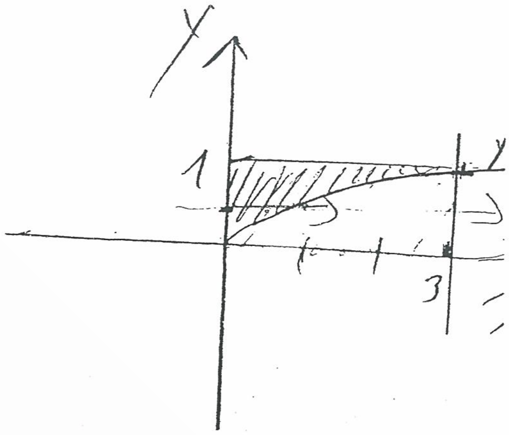
$y = \pm 4 - z$

the pts are $(t, \pm 4 - t, t)$ $t \in \mathbb{R}$

20

5. (20 points) Evaluate the following integrals.

a) $\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\,dy\,dx$

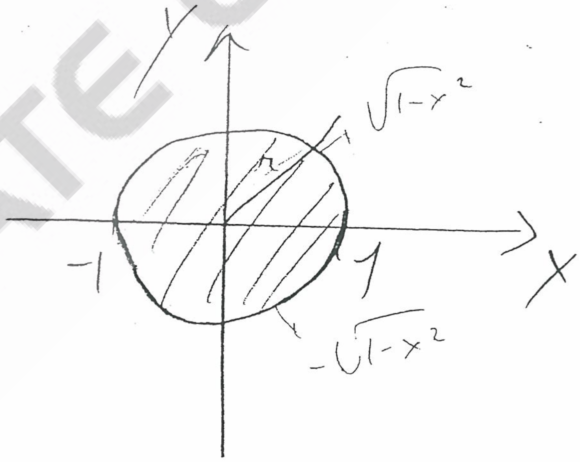


$= \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy = \int_0^1 x e^{y^3} \Big|_0^{3y^2} dy$

$= \int_0^1 3y^2 e^{y^3}\,dy$

$= e^{y^3} \Big|_0^1 = e^1 - e^0 = e - 1$

b) $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,dy\,dx}{\left(1+x^2+y^2\right)^2}$



witching to polar coordinats:

$\int_0^{\pi} \int_0^1 \frac{2r\,dr\,d\theta}{(1+r^2)^2}$

$\int_0^{2\pi} \int_0^1 2r(1+r^2)^{-2}\,dr\,d\theta$

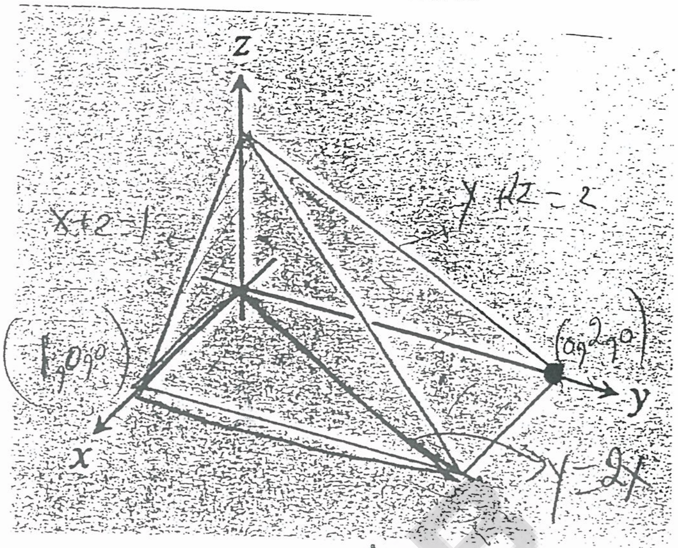
$\int_0^{2\pi} \frac{(1+r^2)^{-1}}{-1} \Big|_0^1 d\theta = \int_0^{2\pi} \frac{-1}{(1+r^2)} \Big|_0^1 d\theta = \int_0^{2\pi} -\frac{1}{2} + 1\,d\theta$

$= \int_0^{2\pi} \frac{1}{2}\,d\theta =$

$= \frac{\theta}{2} \Big|_0^{2\pi} = \pi.$

11

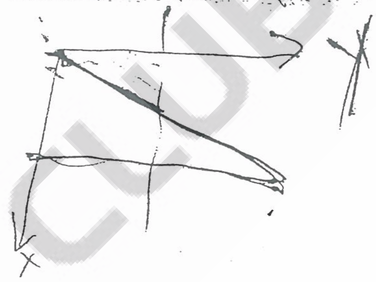
6. **(12 points)** Let D be the region in the first octant bounded by the coordinate planes and the planes $x+z=1$, $y+2z=2$.
Set up triple integrals in rectangular coordinates that gives the volume of D in each of the following orders:



6

a) $dy\,dx\,dz$

$$\int_0^1\int_0^{1-z}\int_0^{2-2z} dy\,dx\,dz$$



5

b) $dz\,dx\,dy$

$$\int_0^2\int_0^{\frac{2-y}{2}}\int_0^{\frac{2-y}{2}} dz\,dx\,dy + \int_0^2\int_{\frac{y}{2}}^{1}\int_0^{1-x} dz\,dx\,dy$$

$$\left(\begin{array}{l} x+z=1 \\ y+2z=2 \end{array}\right\} \quad z=1-x \quad y+2-2x=2 \therefore\ y-2x=0,\ y=2x\Bigg)$$

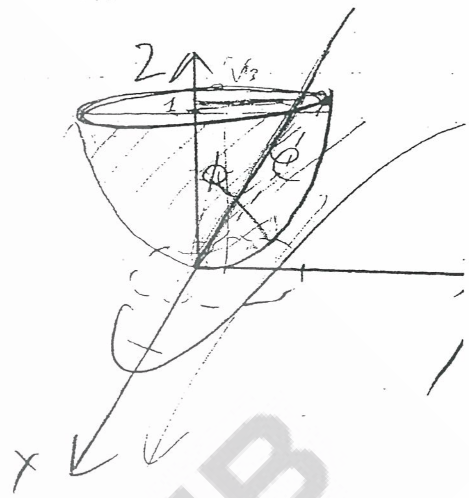
13

7. **(20 points)** Let D be the region bounded below by the paraboloid $z=\frac{x^2}{3}+\frac{y^2}{3}$ and above by the plane $z=1$.

5 a) Set up triple integrals for the volume of D in

i) Rectangular coordinates.

$$V=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{\frac{x^2}{3}+\frac{y^2}{3}}^{1} dz\,dx\,dy$$



$z=z$

$\Rightarrow \frac{x^2}{3}+\frac{y^2}{3}=1$

$\Rightarrow \frac{x^2}{3}+\frac{y^2}{3}=\frac{3}{3}$ circle radius $\sqrt{3}$ origin

3 ii) Cylindrical coordinates.

$$V=\int_{0}^{2\pi}\int_{0}^{\sqrt{3}}\int_{\frac{r^2}{3}}^{1} r\,dz\,dr\,d\theta$$

$\frac{x^2+y^2}{3}=z$

for $z=1$ $r^2=3z=3$

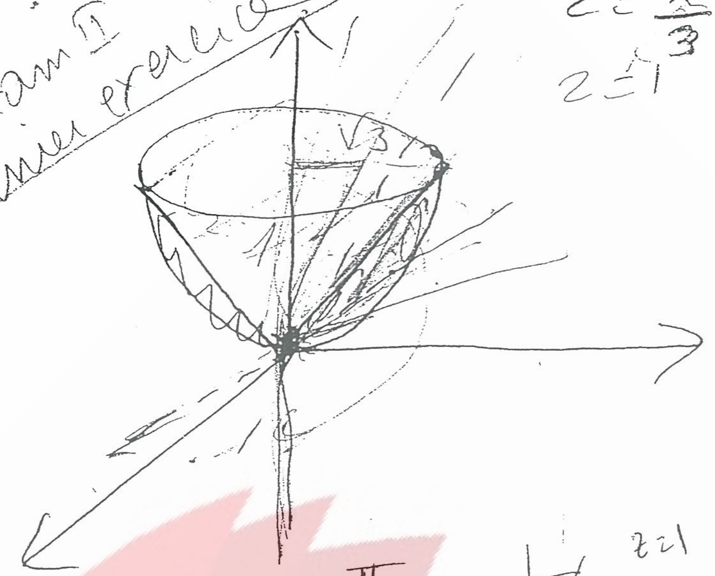
$(r=\pm\sqrt{3})$

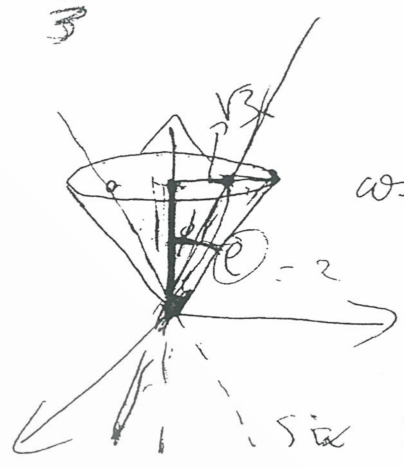
$r=\sqrt{3}$

b) Exam II dernier exercice

$z = \frac{x^2}{3} + \frac{y^2}{3}$

$z = 1$



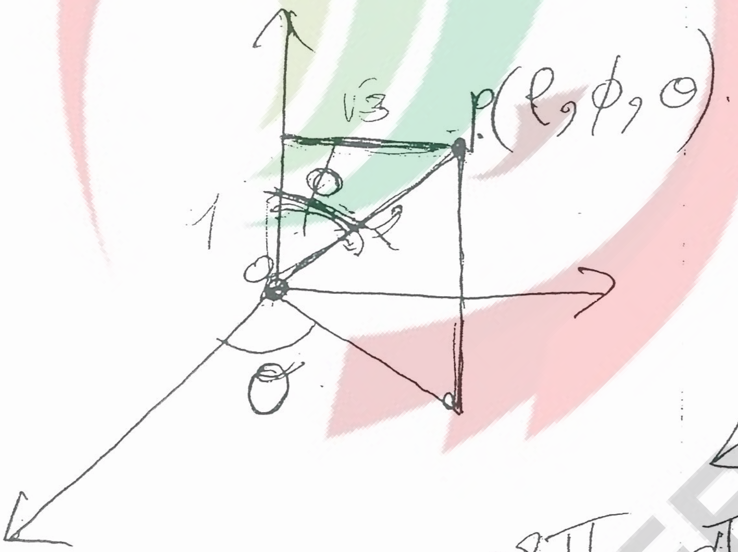


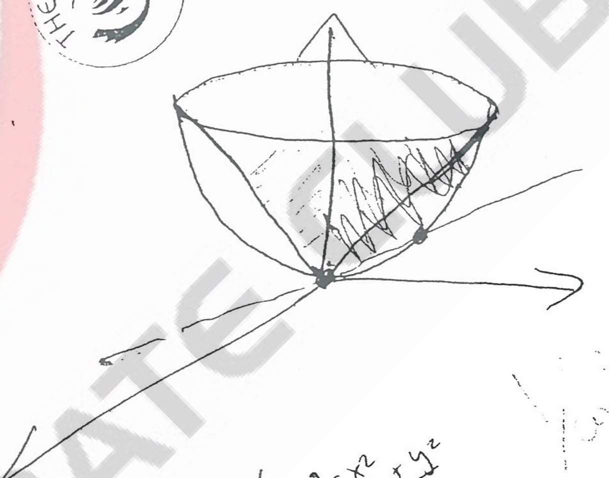
$\cos\phi = \frac{1}{\rho}$

$\sin = \frac{\sqrt{3}}{2}$

$z=1$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{\frac{1}{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$





$z = \frac{x^2}{3} + \frac{y^2}{3}$

$$\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{\frac{3\cos\phi}{\sin^2\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\underbrace{\rho\cos\phi}_{z} = \frac{(\overbrace{\rho\sin\phi\cos\theta}^{x})^2}{3} + \frac{(\overbrace{\rho\sin\phi\sin\theta}^{y})^2}{3}$$

$$\cancel{\rho}\cos\phi = \frac{\rho^{\cancel{2}}}{3}\sin^2\phi \qquad \rho = \frac{3\cos\phi}{\sin^2\phi}$$

4 iii) Spherical coordinates.

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{\frac{\sqrt{3}}{\sin\phi}} \rho \sin\phi \, d\rho \, d\phi \, d\theta$$

$$+ \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{\frac{\sqrt{3}}{\sin\phi}} \text{— —}$$

$\sin\phi = \frac{\sqrt{3}}{\rho}$ (in the figure)

$\sin\phi = \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \therefore \phi = \frac{\pi}{3}$

3 b) Evaluate the volume of D.

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\frac{r^2}{3}}^{1} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} rz \Big|_{\frac{r^2}{3}}^{1} dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(r - \frac{r^3}{3}\right) dr \, d\theta = \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{12} \Big|_0^{\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \frac{3}{2} - \frac{9}{12} \, d\theta$$

$$= \frac{3}{4}\theta \Big|_0^{2\pi} = \frac{3\pi}{2}$$