Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 224
CALCULUS IV
Exam #2
Wednesday June 1st, 2005
Duration: 55 minutes

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Section: ________________

Instructor: Dr. Rached

Grade: 83

Problem Number	Points	Score
1	15	
2	15	
3	10	
4	8	
5	20	
6	12	
7	20	
Total	100	

15

1. (15 points) Test the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points.

$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$

$f_x = 3x^2 + 6x$

$f_y = 3y^2 - 6y$

critical point $f_x = f_y = 0 \Rightarrow 3x^2 + 6x = 0$ $\quad$ $3y^2 - 6y = 0$

$x(3x+6) = 0$ $\quad$ $y(3y-6) = 0$

$\boxed{x=0}$ $\quad 3x+6=0$ $\quad$ $\boxed{y=0}$, $3y=6$

$3x=-6$ $\quad$ $\boxed{y=2}$

$\boxed{x=-2}$

critical points: $(0,0)$, $(-2,2)$, $(0,2)$, $(-2,0)$

$f_{xx} = 6x+6 \Rightarrow f_{xx}$ at $(0,0) = 6 \Rightarrow f_{xx}$ at $(-2,2) = -6$

$f_{yy} = 6y-6. \Rightarrow f_{yy}$ at $(0,0) = -6. \Rightarrow f_{yy}$ at $(-2,2) = 6.$

$f_{xy} = 0.$

at $(0,0)$ - $f_{xx}f_{yy} - f_{xy}^2 = (6)(-6) - 0 = -36 < 0$ $\quad$ $(0,0)$ is a saddle point

at $(-2,2)$ $f_{xx}f_{yy} - f_{xy}^2 = (-6)(6) - 0 = -36 < 0$ $\quad$ $(-2,2)$ is a saddle point

f_{xx} at $(0,2) \Rightarrow f_{xx} = 6$

f_{yy} at $(0,2) \Rightarrow f_{yy} = 6.$

at $(0,2) \Rightarrow$ $f_{xx}f_{yy} - f_{xy}^2 = (6)(6) - 0 = 36 > 0.$

but f_{xx} at $(0,2) = 6 > 0 \Rightarrow (0,2)$ is a local minimum

f_{xx} at $(-2,0) \Rightarrow f_{xx} = -6$

f_{yy} at $(-2,0) \Rightarrow f_{yy} = -6.$

at $(-2,0)$ $f_{xx}f_{yy} - f_{xy}^2 = (-6)(-6) - 0 = 36 > 0$

but f_{xx} at $(-2,0) = -6 < 0 \Rightarrow (-2,0)$ is a local maximum

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- **2. (15 points)** Find the point closest to the origin on the line of intersection of the planes $x+y+z=1$ and $-x+y+z=1$.

$g(x,y,z) = x+y+z-1$

$h(x,y,z) = -x+y+z-1$

point closest to the origin $\Rightarrow$ we have to minimize the equation: $x^2+y^2+z^2 \Rightarrow f(x,y,z) = x^2+y^2+z^2$

$\nabla f = \lambda \nabla g + \mu \nabla h$.

$2x = \lambda - \mu \Rightarrow 2x = \lambda - \mu \Rightarrow x = \frac{\lambda-\mu}{2}$

$2y = \lambda + \mu \Rightarrow y = \frac{\lambda+\mu}{2}$

$2z = \lambda + \mu \Rightarrow z = \frac{\lambda+\mu}{2}$

$\Rightarrow y = z = \frac{\lambda+\mu}{2}$.

$\Rightarrow x + 2y = 1$

$-x + 2y = 1$.

4 $4y = 2$

$\boxed{y=1}$ $\Rightarrow$ $y = z \Rightarrow \boxed{z=1}$

$y = z = \frac{1}{2}$ $x = 0$

$\Rightarrow x + 2 = 1$

$\Rightarrow x = 1-2$

$\Rightarrow \boxed{x=-1}$

point closest to the origin is $(-1, 1, 1)$

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3. **(10 points)** The derivative of $f(x,y)$ at the point $P_0(1,2)$ in the direction of $i+j$ is $2\sqrt{2}$ and in the direction of $-2j$ is -3. What is the derivative of f in the direction of $-i-2j$?

$(D_u f)_{P_0} = \nabla f_{P_0} \cdot u = (f_x, f_y)(1,2) \cdot (i+j)$

$= (f_x i + 2f_y j) \cdot \cancel{(i+j)}$

$= f_x + 2f_y \Rightarrow f_x + 2f_y = 2\sqrt{2}$.

$(f_x, f_y)|_{1,2} \cdot (-2j) = (f_x i + 2f_y j) \cdot (-2j)$

$= -4f_y \Rightarrow -4f_y = -3$

$\Rightarrow f_y = \frac{3}{4}$.

$\Rightarrow f_x + 2\left(\frac{3}{4}\right) = 2\sqrt{2}$

$f_x = 2\sqrt{2} - \frac{3}{2}$

derivative in the direction $(-i-2j)$

$\left(\left(2\sqrt{2} - \frac{3}{2}\right)\vec{i} + \frac{3}{4}\vec{j}\right)\left(-\vec{i} - 2\vec{j}\right) = -2\sqrt{2} + \frac{3}{2} - \frac{3}{2} = -2\sqrt{2}$.

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4. **(8 points)** Find the points on the surface $(y+z)^2 + (z-x)^2 = 16$ where the normal line is parallel to the yz-plane. (Hint: Use Gradient).

$(y+z)^2 + (z-x)^2 = 16$

$\nabla f = -2(z-x),\ 2(y+z),\ 2(x+y+2z)$

normal line at $P(a,b,c)$ — tangent plane:

$x = a - 2(z-x)t$

$y = b + 2(y+z)t$

$z = c + 2(-x+y+2z)t$

$-2z + 2x(x-a) + (2y+2z)(y-b) + (-2x+2y+2z)(y-b)$

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5. (20 points) Evaluate the following integrals.

a) $\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\,dy\,dx$ we reverse the order

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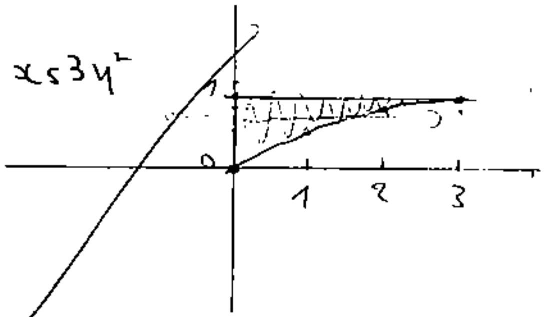
$y = \sqrt{\frac{x}{3}}$

$y^2 = \frac{x}{3} \Rightarrow x = 3y^2$

$\int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy$



$\int_0^1 e^{y^3} x\Big|_0^{3y^2} dx = \int_0^1 e^{y^3}\cdot 3y^2\,dx$.

let $y^3 = t$ $\quad 3y^2\,dy = dt$

$\Rightarrow \int_0^1 e^t\,dt = e^t\Big|_0^1 = e^{y^3}\Big|_0^1$

$= e^1 - e^0$

$= e - 1$

b) $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,dy\,dx}{\left(1+x^2+y^2\right)^2}$

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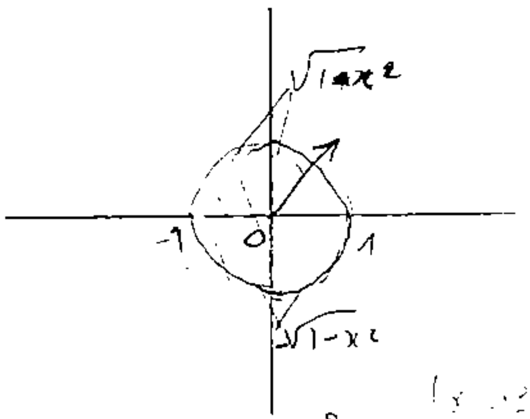
we switch to polar coordinates

$y = \sqrt{1-x^2}$

$y^2 + x^2 = 1$

$\cancel{\int_0^{2\pi}\int_0^1 \frac{2\,dy\,dx}{(1+r^2)^2}}$

$\int_0^{2\pi}\int_0^1 \frac{2r\,dr\,d\theta}{(1+r^2)^2}$ $\qquad 1+r^2 = t$, $\quad 2r\,dr = dt$



$\int_0^{2\pi}\int_0^1 \frac{dt\,d\theta}{t^2} = -\int_0^{2\pi} \frac{1}{1+r^2}\Big|_0^1 d\theta$

$= -\int_0^{2\pi} \left(\frac{1}{1+1}\right) - \left(\frac{1}{1}\right) d\theta = -\int_0^{2\pi} \frac{1}{2} - 1\,d\theta$

$= -\int_0^{2\pi} -\frac{1}{2}\,d\theta = -\left(-\frac{1}{2}\theta\right)\Big|_0^{2\pi}$

$= -\left(-\frac{2\pi}{2} - 0\right) = \pi$

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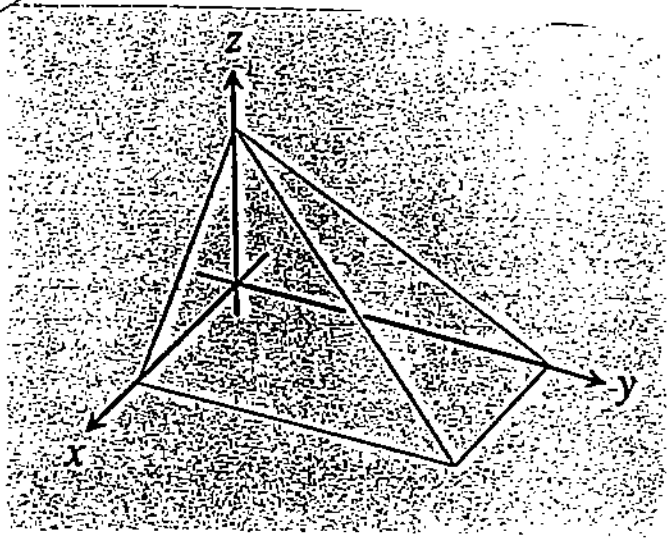
6. **(12 points)** Let D be the region in the first octant bounded by the coordinate planes and the planes $x+z=1$, $y+2z=2$.
Set up triple integrals in rectangular coordinates that gives the volume of D in each of the following orders:



6 a) $dy\,dx\,dz$

$$\int_0^1\int_0^{1-z}\int_0^{2-2z} dy\,dx\,dz$$

$y+2z=2$
$y \le 2-2z$
$x+z \le 1$
$x \le 1-z$

5 b) $dz\,dx\,dy$

$$\cancel{\int_0^2\int_0^{y/2}\int dz\,dx\,dy}$$

$2z \le 2-y$
$z \le 1-\frac{y}{2}$

$$\int_0^2\int_{y/2}^{1}\int_0^{1-x} dz\,dx\,dy + \int_0^2\int_0^{y/2}\int_0^{1-\frac{y}{2}} dz\,dx\,dy$$

$x \le 1-z$
$x \le 1-1+\frac{y}{2}$
$x \le \frac{y}{2} \Rightarrow y \le 2x.$

F4

7. (20 points) Let D be the region bounded below by the paraboloid $z=\frac{x^2}{3}+\frac{y^2}{3}$ and above by the plane $z=1$.

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a) Set up triple integrals for the volume of D in

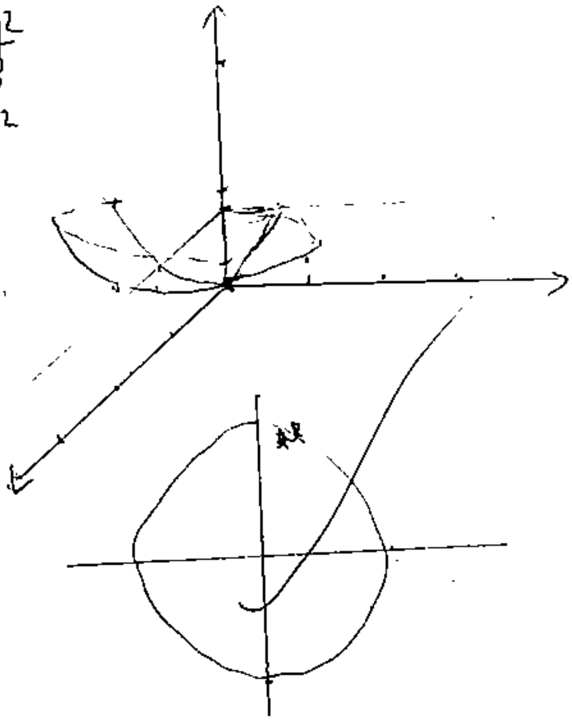
i) Rectangular coordinates.

$$\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{\frac{x^2}{3}+\frac{y^2}{3}}^{1} dz\,dy\,dx$$

$1 = \frac{x^2}{3}+\frac{y^2}{3}$

$3 = x^2+y^2$

$y = \sqrt{3-x^2}$



ii) Cylindrical coordinates.

$$\int_{0}^{2\pi}\int_{0}^{\sqrt{3}}\int_{\frac{r^2}{3}}^{1} r\,dz\,dr\,d\theta$$

$\frac{x^2}{3}+\frac{y^2}{3} = 1$

$x^2+y^2=3$

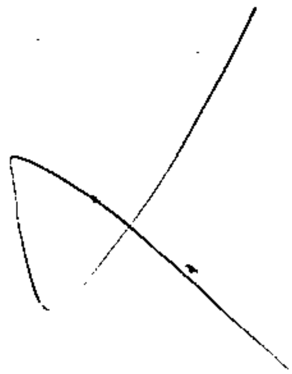
$R=\sqrt{3}$

2 iii) Spherical coordinates.

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\frac{\sqrt{3}}{\sin\phi}} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$\rho = \frac{r}{\sin\phi}$ $\quad r = \sqrt{3}$

$\rho = \frac{\sqrt{3}}{\sin\phi}$



b) Evaluate the volume of *D*.

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$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\frac{r^2}{3}}^{1} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} rz\Big|_{\frac{r^2}{3}}^{1}$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r - \frac{r^3}{3}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{12}\Big|_0^{\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \frac{3}{2} - \frac{9}{12}\, d\theta$$

$$= \int_0^{2\pi} \frac{18-9}{12}\, d\theta = \int_0^{2\pi} \frac{9}{12}\, d\theta = \frac{9\theta}{12}\Big|_0^{2\pi}$$

$$= \frac{18\pi}{12} = \frac{9\pi}{6} = \frac{3\pi}{2}$$