1) **(15 points)** Evaluate the integral $I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} \frac{x+y}{x^2+y^2}\, dy\,dx$.

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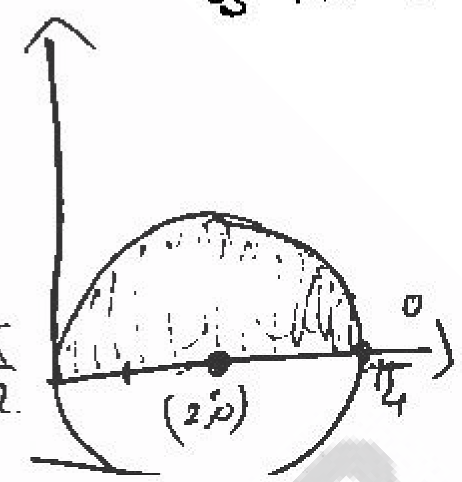
$\int_0^4 \int_0^{\sqrt{4-(x-2)}} \frac{x+y}{x^2+y^2}\, dy\,dx$

let us change it to polar coordinates. $y^2+(x-2)^2=4$.



$\Rightarrow I = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{4-(r\cos\theta-2)^2}} \frac{r(\cos\theta+\sin\theta)}{r^2} \times r\, dr\,d\theta$

$y^2 + x^2 - \overset{4}{2}x + 4 = 4$.

$\Rightarrow r^2 - \overset{4}{2}r\cos\theta = 0$

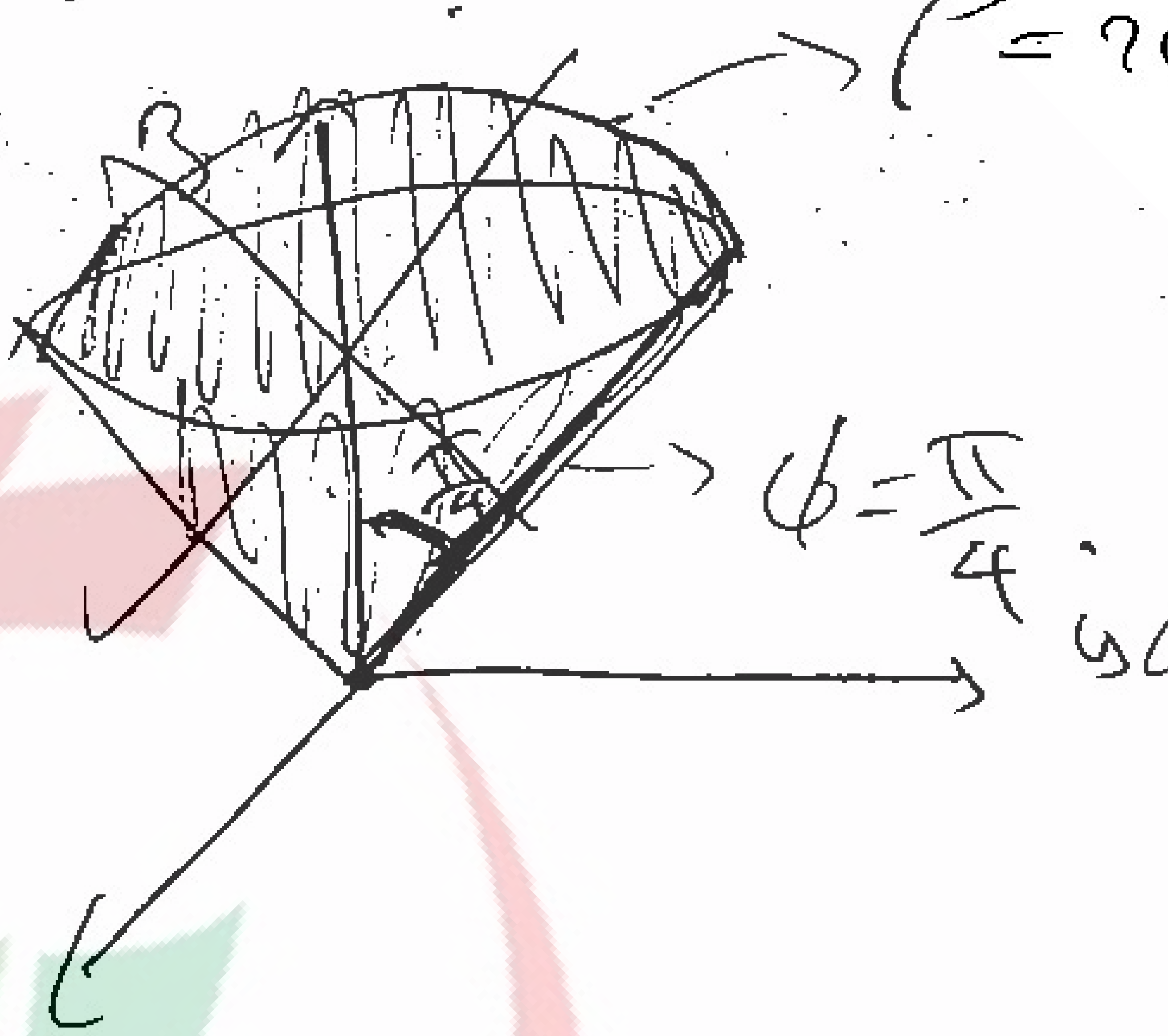
$\Rightarrow r = \overset{4}{2}\cos\theta$.

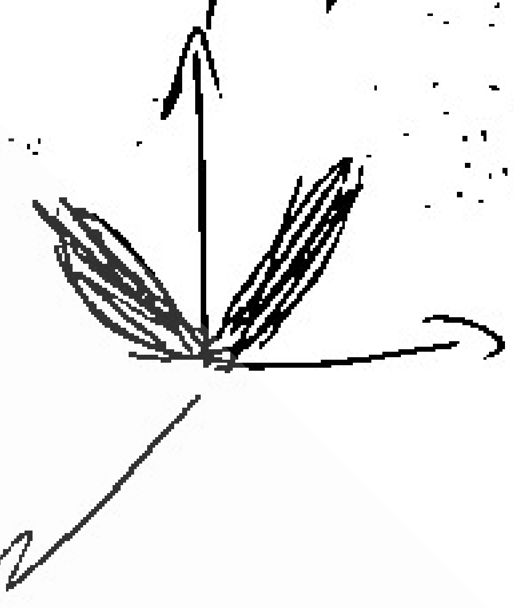
-1 $\Rightarrow I = \int_0^{\frac{\pi}{2}} \int_0^{\overset{4}{2}\cos\theta} (\cos\theta+\sin\theta)\, dr\,d\theta$ follow

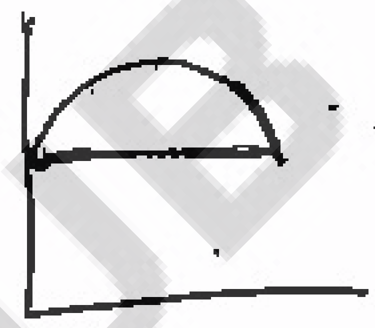
-1 $\Rightarrow I = \int_0^{\frac{\pi}{2}} (2\cos^2\theta + 2\cos\theta\sin\theta)\, d\theta = \int_0^{\frac{\pi}{2}} (1+\cos 2\theta + \sin 2\theta)\, d\theta$ follow

-1 $= \left[\theta - \frac{1}{2}\sin 2\theta - \frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} + \frac{1}{2}\cancel{\left(-\frac{1}{2}\right)} = \boxed{\frac{\pi}{2}}$

2) **(15 points)** Find the volume of the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z=\sqrt{x^2+y^2}$.







cone: $z=\sqrt{x^2+y^2} = \sqrt{r^2}$ $\quad r = \rho\sin\phi$.

$\Rightarrow z = \rho\cos\phi = \sqrt{(\rho\sin\phi)^2}$

$\Rightarrow \cos\phi = \sin\phi \Rightarrow \boxed{\phi = \frac{\pi}{4}}$.

following: $\Rightarrow$ Volume $= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\cos\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$.

OK

$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3}\sin\phi\right]_0^{2\cos\phi} d\phi \, d\theta$.

$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\frac{8}{3}\cos^3\phi\sin\phi\right) d\phi \, d\theta$.

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$u' \; u^n$

$= \int_0^{2\pi} -\frac{8}{3}\left[\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{4}} d\theta$.

$= \int_0^{2\pi}\left(-\frac{1}{6} + \frac{8}{12}\right) d\theta = \int_0^{2\pi} \frac{6}{12} \, d\theta = \left[\frac{1}{2}\theta\right]_0^{2\pi}$

$= \pi$ ok u^3.

3) **(15 points)** Evaluate the integral $K=\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx \quad (b>a>0)$.

Hint: $\frac{e^{-ax}-e^{-bx}}{x}=\int_a^b e^{-xy}dy$.

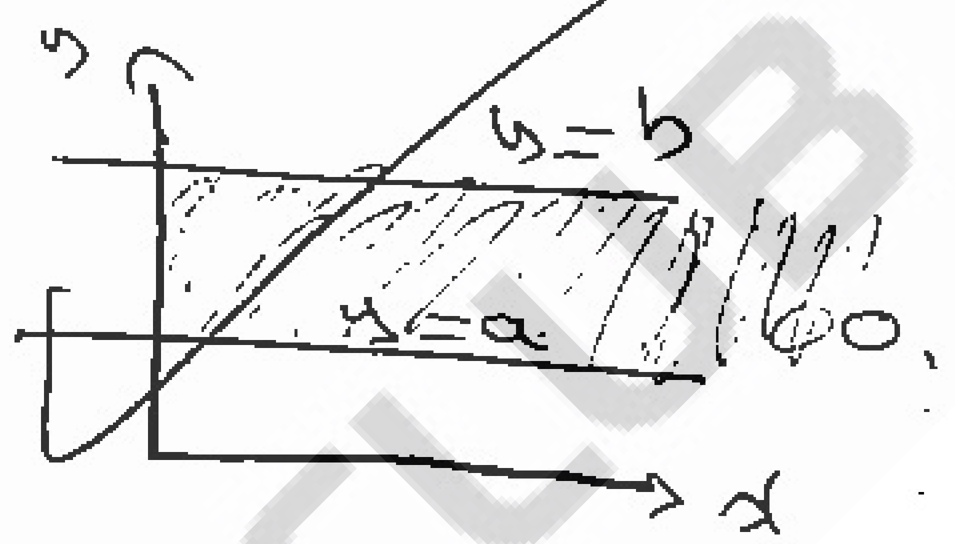
$$K=\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}dx=\int_0^{\infty}\int_a^b e^{-xy}dy\,dx$$

$$\frac{e^{-ax}-e^{-bx}}{x}=\int_0^b e^{-xy}dy$$

$$\Rightarrow \int_0^{\infty}\int_a^b e^{-xy}dy\,dx$$

$y=a$

$y=b$



let us reverse the order of integration.

$$\Rightarrow \int_a^b\int_0^{\infty} e^{-xy}dx\,dy.$$

$$\Rightarrow \int_a^b -\frac{1}{y}\int_0^{\infty} -y\,e^{-xy}dx\,dy.$$

$$=\int_a^b -\frac{1}{y}\left[e^{-xy}\right]_0^{\infty}dy.$$

$$=\int_0^b -\frac{1}{y}\times\left[0-1\right]dy=\int_a^b\frac{1}{y}\,dy.$$

$$=\left[\ln y\right]_a^b=(\ln b-\ln a)$$

$b>a>0.$

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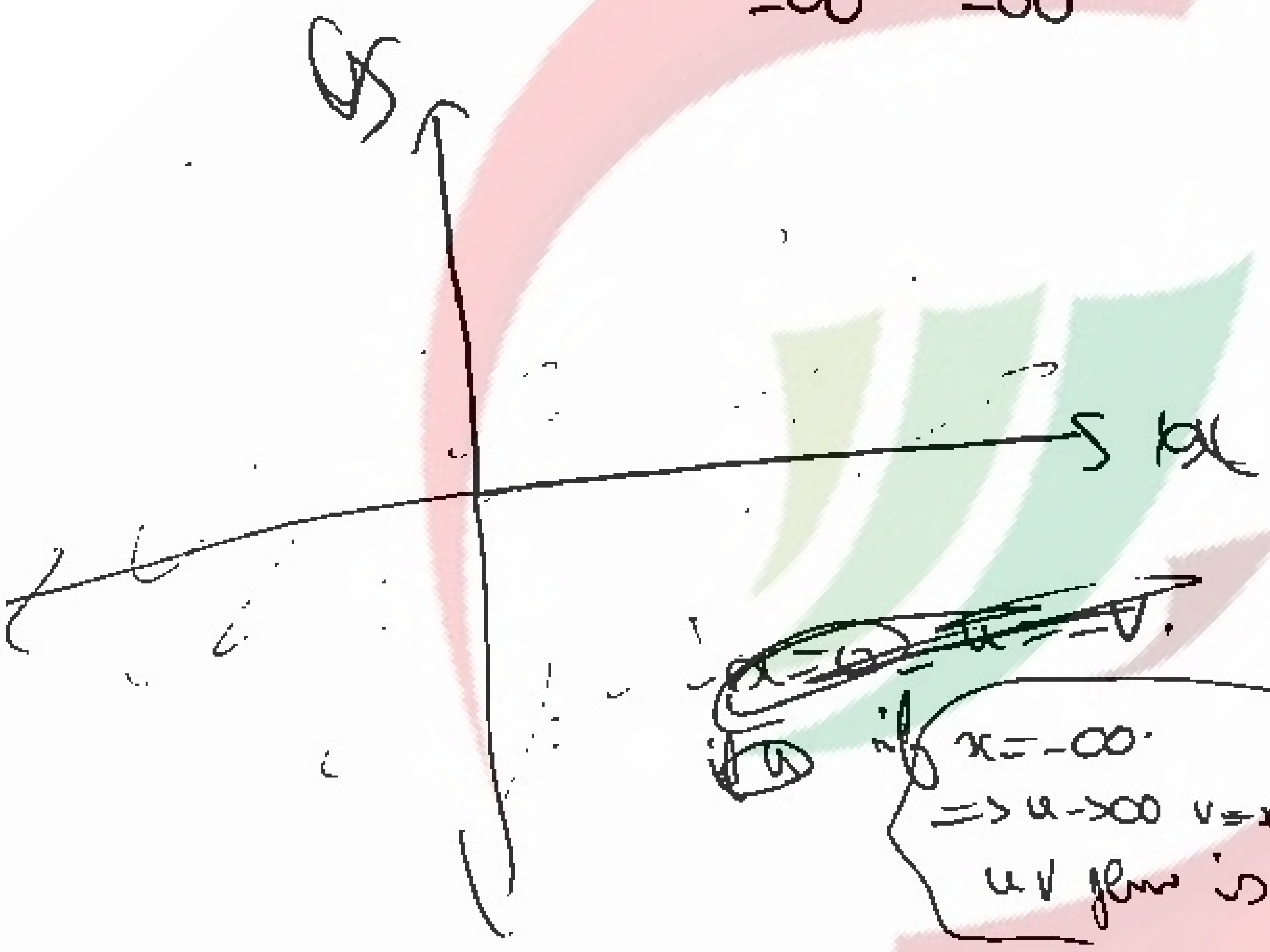
4) (15 points) It is well-known that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Evaluate the integral $A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}dxdy$ by using the transformation $x = u+v$, $y = -u+v$.

Hint: The uv-domain of integration is the whole uv-plane.

$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}\,dx\,dy.$$



let $x = u+v.$
$y = -u+v$

~~x = u - v.~~

if $x = -\infty$
$\Rightarrow u \to \infty \quad v = \infty$
uv plane is the whole [illegible]

$$\Rightarrow J(x,y) = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

$$\Rightarrow A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}\,dx\,dy = \int\int e^{-(u^2+v^2+2uv+\dots)}$$

x^2+xy+y^2 if $x = u+v$. $y = v-u$ =

$u^2+v^2+2uv+(u+v)(v-u)+v^2+u^2-2uv$

$2u^2+2v^2+4uv-u^2+v^2-vu = \boxed{u^2+3v^2}$

$$\Rightarrow A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-3v^2} \times |J(u,v)|\,du\,dv.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2e^{-u^2-3v^2}\,du\,dv.$$

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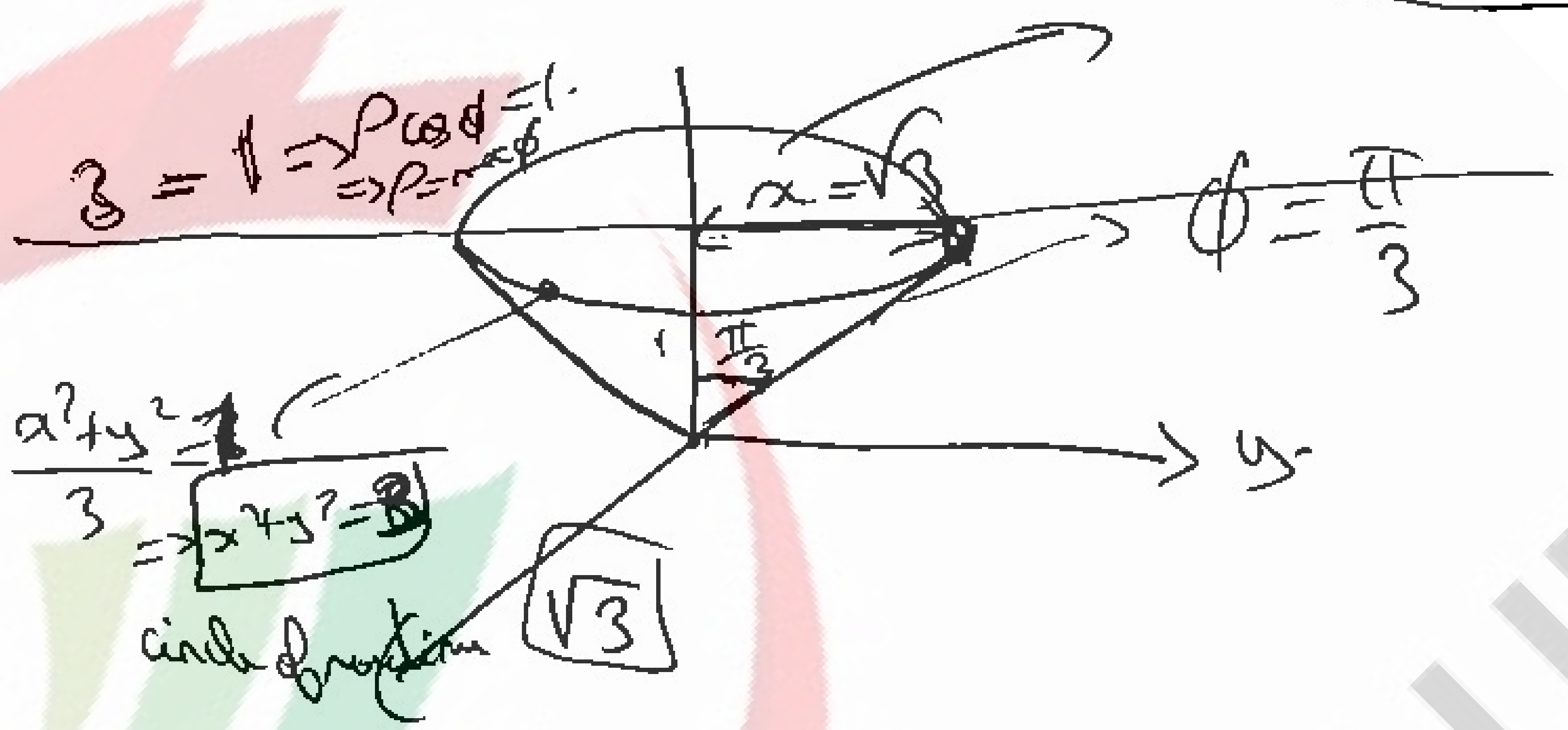
5) **(20 points)** Let D be the solid bounded below by the cone $\phi = \frac{\pi}{3}$ and above by the plane $z = 1$. Set up (but do not evaluate) triple integrals for the volume of D in

a) Spherical coordinates, using the order $d\rho\, d\phi\, d\theta$.
b) Cylindrical coordinates, using the order $dz\, dr\, d\theta$.
c) Rectangular coordinates, using the order $dz\, dy\, dx$.

$\frac{x}{1} = \tan\frac{\pi}{3} \Rightarrow \boxed{x = \sqrt{3}}$



$\phi = \frac{\pi}{3} \Rightarrow$

$\cos\phi = \frac{1}{2}$

$\Rightarrow \frac{3}{\rho} = \frac{1}{2}$

a) Spherical coordinates: $\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

b) cylindrical coordinates: at $z = 1$ intersect the cone as the circle.

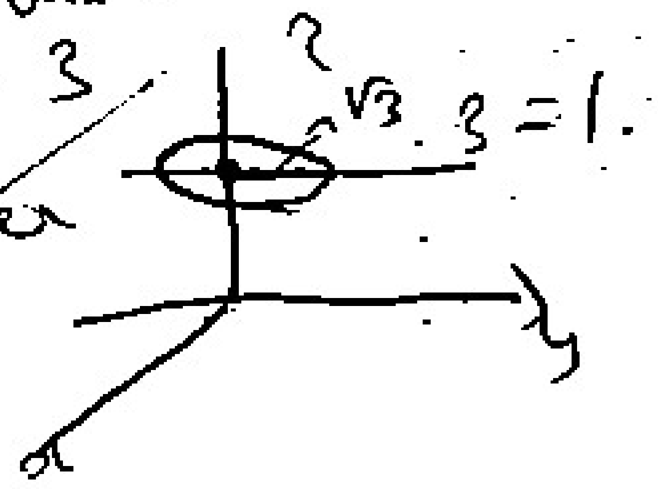
$\int_0^{2\pi}\int_0^{r=\sqrt{3}}\int_{z=\frac{r}{\sqrt{3}}}^{1} dz\, dr\, d\theta$ $\quad \boxed{x^2+y^2 = 3}$ $\boxed{\text{at } z=1 \Rightarrow r^2 = 3 \Rightarrow r = \sqrt{3}}$

$z = \sqrt{\frac{r^2}{3}}$

$x^2+y^2 = 3$

$\Rightarrow$ the formula of the cone is

$\boxed{z^2 = \frac{x^2+y^2}{3}}$

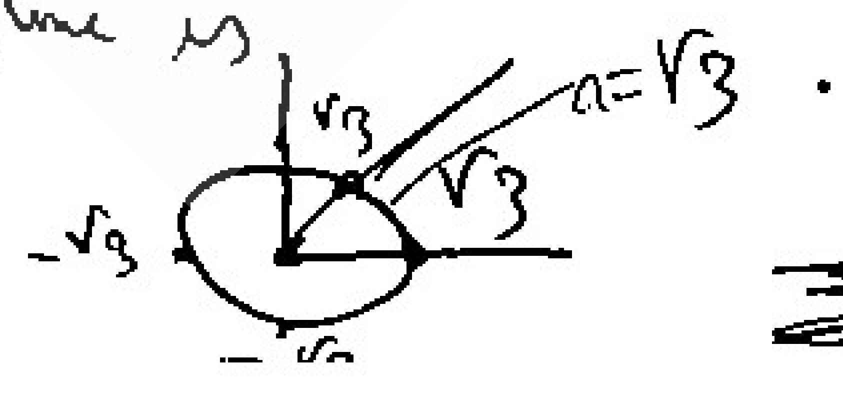


verification

$\rho^2\cos^2\phi = \frac{\rho^2\sin^2\phi}{3}$

$\Rightarrow \frac{\sin^2\phi}{\cos^2\phi} = 3 \Rightarrow \boxed{\tan\phi = \sqrt{3}}$

Projection at xy plane is



6) (20 points) Find the points on the curve $x^2 + xy + y^2 = 1$ that are nearest to and farthest from the origin.

orig let $f(x,y) = x^2 + y^2 + \cancel{z^2}$

on $g(x,y) = x^2 + xy + y^2 - 1$.

$\Rightarrow \begin{cases} \Delta f(x,y) = \lambda \Delta g(x,y) \\ g(x,y) = 0 \end{cases} = \begin{cases} 2x\,i + 2y\,j = (2x+y)\,i + (x+2y)\,j \\ x^2 + xy + y^2 - 1 = 0 \end{cases}$

$\Rightarrow \begin{cases} 2x = 2\lambda x + y\lambda & (1) \\ 2y = \lambda x + 2y\lambda & (2) \end{cases}$

$y + x = 0$

$x = \frac{2y}{\frac{y}{1-2}}$ $\quad x^2 - x^2 \ldots = 1$

$y(2-2\lambda)$

$(1,-1)$

$(-1,1)$

from the (1) : $x = \frac{y\lambda}{2-2\lambda}$

from the 2 : $x = \frac{2y - 2y\lambda}{\lambda}$ $\Rightarrow \frac{y\lambda}{2-2\lambda} = \frac{2y - 2y\lambda}{\lambda}$

$y(2-2\lambda)^2 = y\lambda^2$

$\Rightarrow y\lambda^2 = 4y - 4y\lambda - 2y\lambda + 4\lambda^2 y$ $\quad 12-16+4$

$\Rightarrow 3\lambda^2 y - 6\lambda y + 4y = 0$ $\quad y(4 + 4\lambda^2 - 8\lambda - \lambda$

$y(3\lambda^2 - 8\lambda + 4)$

$2 \quad \frac{2}{3}$

$\Rightarrow y(3\lambda^2 - 6\lambda + 4) = 0$

$\Rightarrow \boxed{y = 0} \quad \boxed{3\lambda^2 - 6\lambda + 4 = 0}$

if $y = 0$ $\quad \frac{3}{9} - \frac{2}{3} + 2$

from (1) $x = 0$. let's put it in the constraint $\quad -\frac{21+45}{9}$

we got $g(0,0) = 0 + 0 + 0 - 1 = 0 \neq$ rejected.

$\frac{21}{9} - \frac{12}{3} + 4 \quad \frac{3}{9} - \frac{6}{3} + 4$ $\quad \Rightarrow$