1) **(24 points)** For each of the four multiple-choice questions below, circle the letter of the correct answer. If more than one letter is circled in the same problem, you will receive no credit for that problem.

Question A

If $2xz + e^{y+z} - \ln(xy) = 2$ defines z implicitly as a differentiable function of x and y, then $\frac{\partial z}{\partial x}$ at $(1, \ln 2, 0)$ will have the value:

a) $\frac{1}{4}$ b) $\frac{1}{2}$ c) $-\frac{1}{4}$ d) $-\frac{1}{2}$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(2z - \frac{1}{x})}{(2x + e^{y+z})} = \frac{-(2\times 0 - \frac{1}{1})}{2\times 1 + e^{\ln 2 + 0}} = \frac{+1}{4}$$

Question B

If $f(x,y) = x^2 y + x\cos y$, then the largest value of the directional derivative at P (1, 0) is equal to:

a) 1 b) $\sqrt{2}$ c) $\sqrt{17}$ d) $\sqrt{5}$

$\nabla f = (2xy + \cos y)i + (x^2 - x\sin y)j$ at $P(1,0)$

$\nabla f = 1\vec{i} + 1\vec{j}$ $\quad \|\nabla f\| \times \cos 0 = \sqrt{2} \times 1 = \sqrt{2}$

Question C

An equation for the plane tangent to the surface $z = x^2 - 4y^2 - 3$ at P (2, 1, 5) is:

a) $4x - 8y = 0$
b) $4x + 8y - z = 11$
c) $4x + 8y = 16$
d) $4x - 8y - z = -5$

$\nabla f = 2x\,i - 8y\,j + 0\vec{K}$

at $P(2,1,5) \Rightarrow 4\vec{i} - 8\vec{j} + 0\vec{K}$

eq. of tan plan

$4(x-2) + (-8)(y-1) + 0(z-5) = 0$

$4x - 8 - 8y + 8 = 0 \rightarrow 4x - 8y = 0$

Question D

Consider the function $f(x,y)$ such that $f_x = 9 - 9x^2$ and $f_y = 2y + 4$. Then $f(x,y)$ has:

a) a local maximum at (1,-2) and a saddle point at (-1,-2);
b) a local minimum at (1,-2) and a saddle point at (-1,-2);
c) a saddle point at (1,-2) and a local minimum at (-1,-2);
d) a saddle point at (1,-2) and a local maximum at (-1,-2).

$f_x = 0 \;;\; 9 = 9x^2$
$x^2 = 1$
$x = \mp 1$

$f_y = 0 \;;\; 2y = -4$
$y = -2$

critical pts $(1,-2) \;;\; (-1,-2)$

$f_{xx} = -18x$

$f_{yy} = 2$

$f_{yx} = 0$

f_{xx} at $(-1,-2) = 18 > 0$
$f_{yy} > 0$ then at $(-1,-2)$ minimal
$\Delta = (-18x)2 - 0$
at $(1,-2)$; $\Delta = -36 < 0$ at $(-1,-2)$

2) (10 points) Consider the function $f(x,y,z)=\ln(x^2+4y^2-z^2)$.

a)(5 points) Find an equation for the level surface of this function that passes through the point (2,1,2).

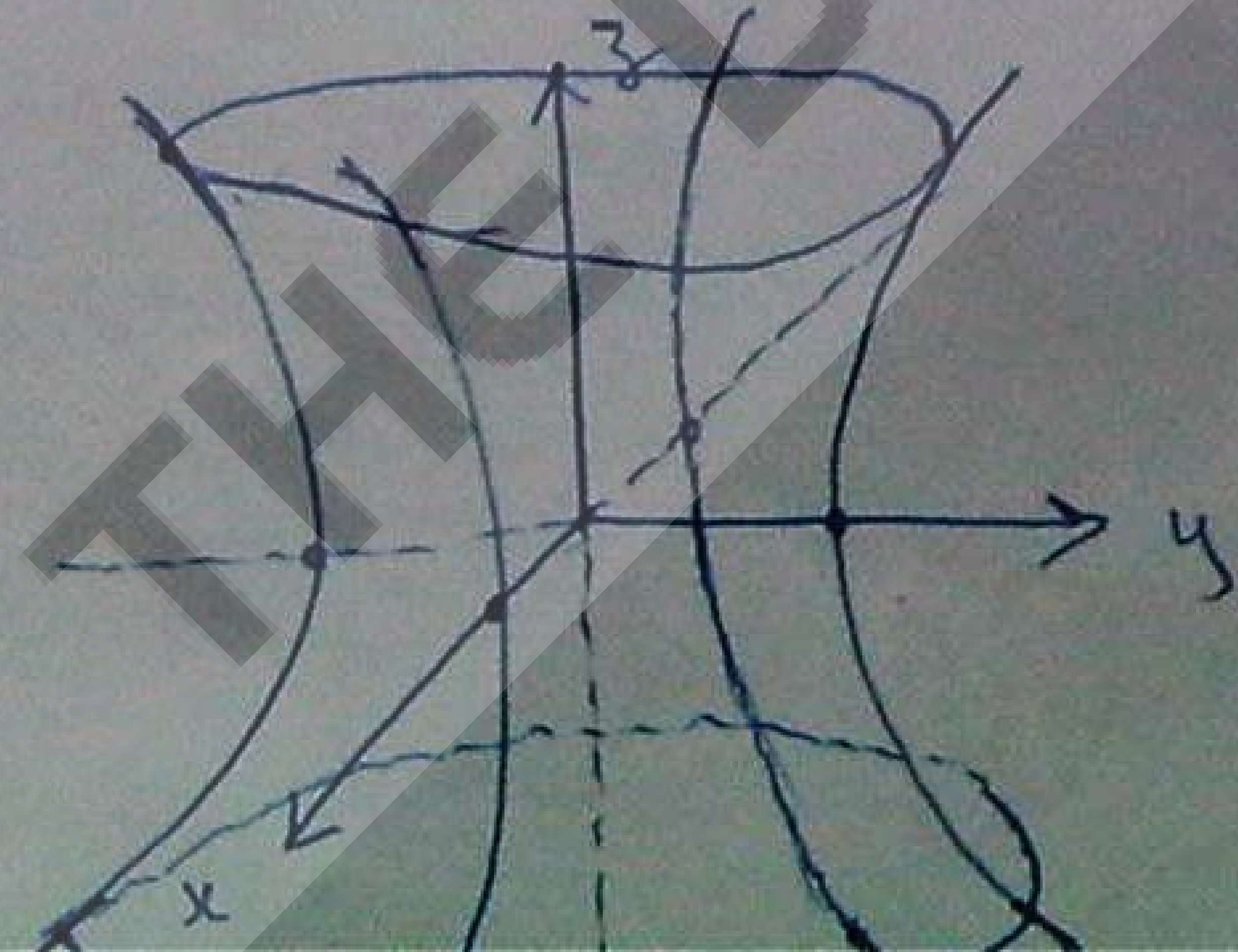
$$c = \ln(x^2+4y^2-z^2)$$
$$c = \ln(4+4-4)$$
$$c = \ln 4$$
$$\ln 4 = \ln(x^2+4y^2-z^2)$$
$$4 = x^2+4y^2-z^2$$

5

b) (5 points) Identify and sketch this level surface.

$$c = \ln 4 = \ln(x^2+4y^2-z^2)$$
$$4 = x^2+4y^2-z^2$$
$$\frac{x^2}{4}+y^2-\frac{z^2}{4}=1 \quad \text{it's a hyperboloid of 1 sheet}$$
$$x=0;\ y^2-\frac{z^2}{4}=1 \quad \text{hyperbola in } yz\text{-plane}$$
$$y=0;\ \frac{x^2}{4}-\frac{z^2}{4}=1 \quad \text{hyperbola in } xz\text{-plane}$$
$$z=0;\ \frac{x^2}{4}+y^2=1 \quad \text{ellipse in } xy\text{-plane}$$

4



3) (19 points) Let $f(x,y)=\sqrt{36-9x^2-4y^2}$

a) (3 points) Find the function's domain and graph it clearly

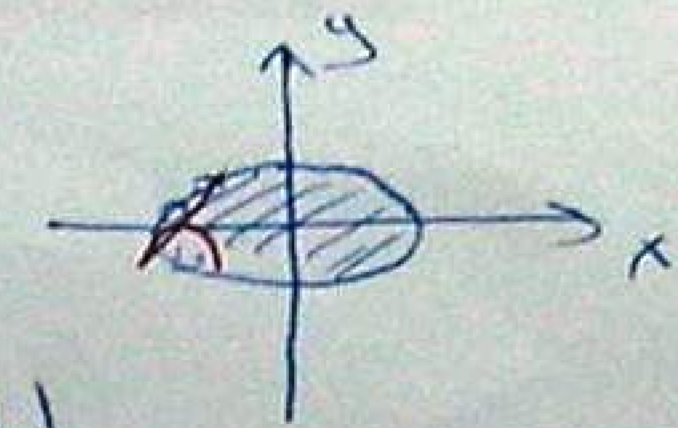
$36-9x^2-4y^2 \geq 0$

$9x^2+4y^2 \leq 36$

$\frac{x^2}{4}+\frac{y^2}{9} \leq 1$

ellipse all pts inside the ellipse $\frac{x^2}{4}+\frac{y^2}{9} \leq 1$

2



b) (2 points) Determine the boundary of the domain

boundary of the domain is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

2

c) (4 points) Is the domain closed, open, or neither? Justify your answer

domain is closed because it contains all its boundary pts

4

d) (2 points) Is the domain bounded or unbounded? Justify your answer

bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

1

d) (3 points) Find the function's range

the range is all positive numbers

$\Rightarrow$ Range is $\mathbb{R}^+$

e) (5 points) Describe the function's level curves, and sketch a few of them.

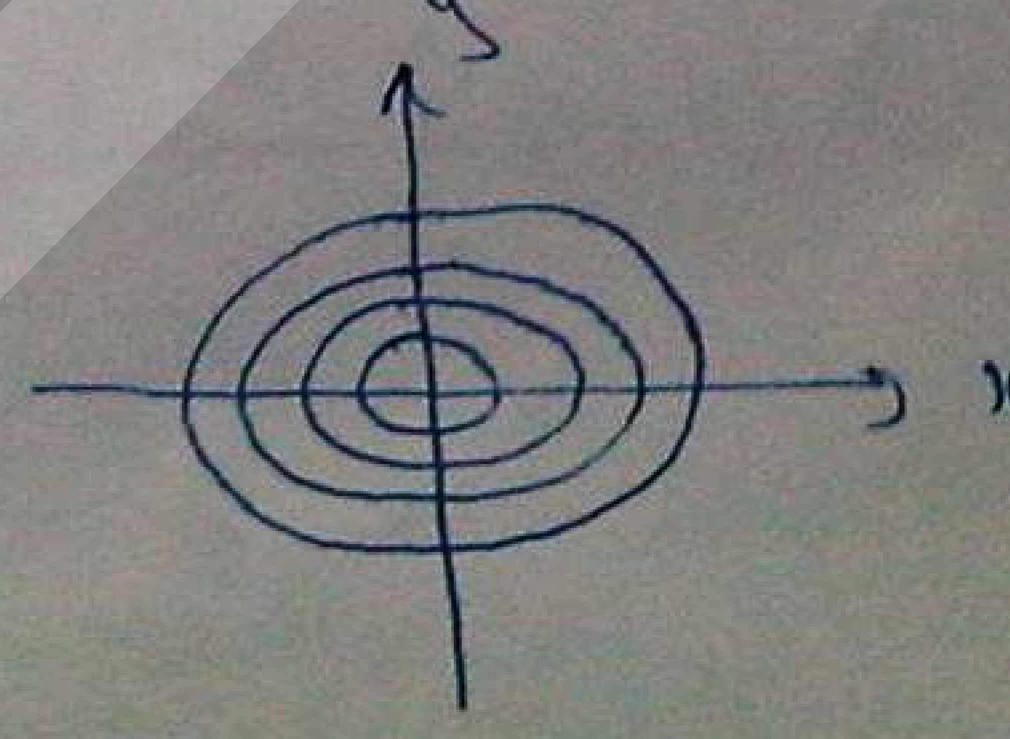
$c=\sqrt{36-9x^2-4y^2}$

$c^2=36-9x^2-4y^2$

$9x^2+4y^2=36-c^2$

$\frac{9x^2}{36-c^2}+\frac{4y^2}{36-c^2}=1$

its the eq of an ellipse in xy plane



4.

4) **(13 points)** Answer each of the following questions:

a) **(5 points)** Consider the function $f(x,y)=\dfrac{x^3y}{x^6+y^2}$. Find the limit of this function as $(x,y)\to(0,0)$ or prove that this limit does not exist.

$$f(x,y)=\frac{x^3y}{x^6+y^2} \qquad \text{let } y=Kx^3$$

$$\lim_{(x,y)\to(0,0)}\frac{x^3y}{x^6+y^2}=\frac{0}{0} \Rightarrow \lim_{(x,y)\to(0,0)}\frac{x^3(Kx^3)}{x^6+(Kx^3)^2}$$

$$=\lim_{(x,y)\to(0,0)}\frac{x^6K}{(K^2+1)x^6}=\lim_{(x,y)\to(0,0)}\frac{K}{K^2+1}\quad \begin{cases} K=0 & \lim \frac{K}{K^2+1}=0\\ K=1 & \lim\frac{K}{K^2+1}=\frac{1}{2}\\ \text{2 different limits} \end{cases}$$

4

then this limit does not exist. (by the path test)

different

b) **(8 points)** If possible, define $f(0,0)$ in a way that extends f to be continuous at the origin, where $f(x,y)=e^{\left(\frac{2x^2y^2}{x^2+y^2}\right)}$.

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim e^{\left(\frac{2x^2y^2}{x^2+y^2}\right)}$$

let $y=Kx$

$$\lim e^{\left(\frac{2x^2K^2x^2}{x^2+K^2x^2}\right)}=\lim e^{\frac{2x^4K^2}{x^2(1+K^2)}}=\lim_{(x,y)\to(0,0)} e^{\left[\frac{2x^2K^2}{(1+K^2)}\right]}$$

$$=e^{\left(\frac{2\times(0)\times K^2}{(1+K^2)}\right)}=e^0=1$$

limit exist and function $f(x,y)$ is defined

5) **(18 points)** Assume the equations: $x = u^2 - v^2$, $y = ve^u$ define u and v as differentiable functions of the two independent variables x and y

a) (12 points) Express u_y and v_y in terms of u and v

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial y} \quad ; \quad \frac{\partial x}{\partial y} = \frac{\partial x}{\partial u} \times \frac{\partial u}{\partial y} = 2u \times \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \times 2u \times \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2u}$$

2

$$2u \times u_y + -2v\, v_y = u$$

$$ve^u \times u_y + e^u v_y = v$$

b) (6 points) Let $s = v\ln u$. Express $\frac{\partial s}{\partial y}$ in terms of u and v

2

$$\frac{\partial s}{\partial y} = \frac{\partial v}{\partial y} \ln u + v \frac{\partial u}{\partial y} \times \frac{1}{u}$$

$$\frac{\partial s}{\partial y} = v_y \ln u + \frac{v}{u} u_y \quad \text{~~and u_y and~~}$$

v_y and u_y are in term of u and v from part (a)

the $\frac{\partial s}{\partial y}$ will be expressed in term of u and v

6) (9 points) Estimate how much the value of $f(x,y,z) = x^2y - y^2z$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(2,2,3)$ straight toward $P_1(3,4,5)$.

ds = 0.1 unit

the change is equal to : $\|\nabla f\| \times \|u\| \times ds$

$$\nabla f = 2xy\,i + (x^2 - 2zy)\,j - y^2\,\vec{k}$$

$$\cancel{\text{at } P_0 ,\ \nabla f = \ldots\vec{i} - 31\vec{j} - 16\vec{k}}$$

$$\cancel{\|\nabla f\| = \sqrt{24^2 + (31)^2 + (-16)^2} = 49.34}$$

at P_0 ; $\nabla f = 8\vec{i} - 8\vec{j} - 4\vec{k}$

$$\|\nabla f\| = \sqrt{8^2 + (8)^2 + (-4)^2} = \sqrt{144} = 12$$

$P_1P_0\,(1, 2, 2)$

$$\|P_1P_0\| = \sqrt{1+4+4} = 3$$

$$\vec{u} = \frac{P_1P_0}{\|P_1P_0\|} = \frac{1i + 2j + 2k}{3} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\|\vec{u}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{1} = 1$$

therefor the change is equal to :

$$Df = 12 \times 1 \times 0.1 = 1.2 \text{ unit}$$

7) (7 points) Let $f(x,y)=x^2y^3+3xy+4y$. Find f_x using the definition

$$\boxed{f_x = 2y^3x + 3y}$$

using definition:

$$f_x = \frac{\partial u}{\partial x} = \lim$$