Exam I (78)

1) **(16 points)** Let $f(x,y)=\sqrt{x^2-y}$.

a) **(2 points)** Find the function's domain.

$x^2 - y \geq 0$

$-y \geq x^2$

$y \leq x^2$

$R^2 - \{y > x^2\}$

b) **(2 points)** Find the function's range.

Range $[0, \infty[$

c) **(4 points)** Describe the function's level curves, and sketch a few of them.

$\sqrt{x^2-y} = c$

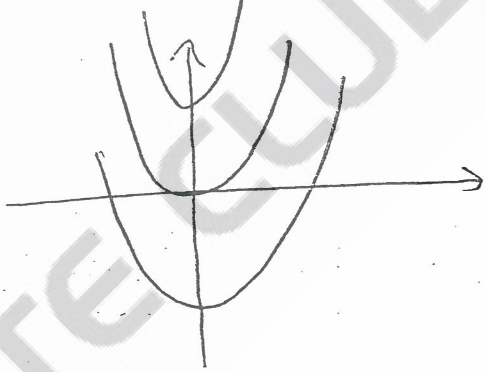
$c^2 = x^2 - y$

parabolas

$x^2 - y = c^2$

$\frac{x^2}{c^2} - \frac{y}{c^2} = 1$

-2



d) **(2 points)** Find the boundary of the function's domain.

~~unbounded~~

$y = x^2$

e) **(4 points)** Determine if the domain is an open region, a closed region or neither, and explain why.

~~open~~ $\boxed{\text{closed}}$

-3

f) **(2 points)** Decide if the domain is bounded or unbounded.

~~unbounded~~

unbaded

-7

2) **(10 points)** Let $f(x,y,z) = x^2 - \frac{y^2}{4} - \frac{z^2}{9}$.

a) **(5 points)** Find an equation for the level surface of this function through the point $\left(\sqrt{2}, 2, 3\right)$.

$x^2 - \frac{y^2}{4} - \frac{z^2}{9} = c$

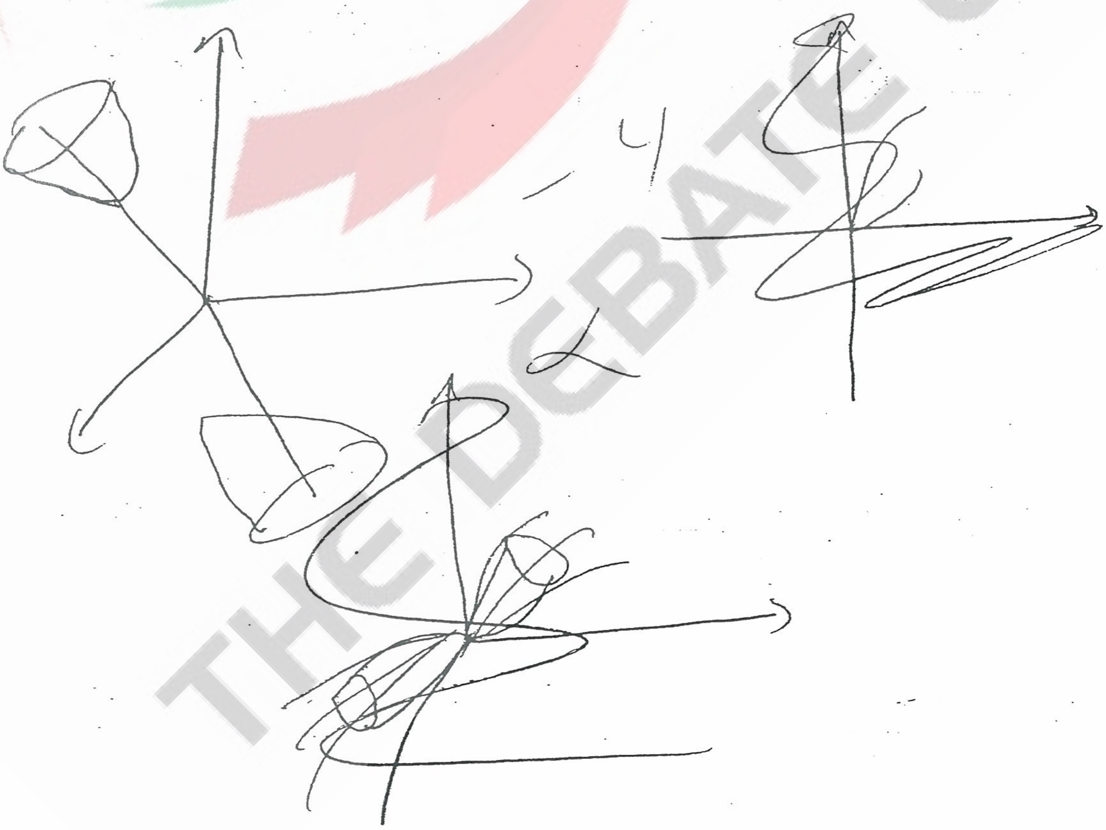
$2 - \frac{4}{4} - \frac{9}{9} = c$

$2 - 1 - 1 = c$

$\boxed{c = 0}$

$x^2 - \frac{y^2}{4} - \frac{z^2}{9} = 0$

b) **(5 points)** Sketch this level surface.



-4

-4

3) (13 points)

a) **(5 points)** Find the following limit or show that it does not exist: $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$.

$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ let $y = mx^2$

$\lim_{x\to 0} \frac{x^2 + mx^2}{mx^2} = \lim \frac{x^2(1+m)}{x^2 m} = \lim \frac{1+m}{m}$ Limit doesn Exist becau it dempend o m.

b) **(8 points)** Let $f(x,y) = \cos\left(\frac{x^3-y^3}{x^2+y^2}\right)$. If possible, define $f(0,0)$ in a way that extends f to be continuous at the origin.

$x = r\cos\alpha$ $r = x^2+y$
$y = r\sin\alpha$

$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right) =$

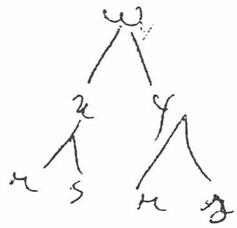
$\lim_{r\to 0} \cos\left(\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r}\right) = \lim_{r\to 0} \cos\left(\frac{r^3(\cos^3\theta - \sin^3\theta)}{r}\right)$

$\lim_{r\to 0} \cos\left(r^2(\cos^3\theta - \sin^3\theta)\right) = \lim_{r\to 0} \cos 0 = 1$

$f(0,0) = 1$

4) **(15 points)** Let $w=f(x,y)$ be a function of x and y where $x=\frac{r}{s}$ and $y=rs$.

a) **(8 points)** Show that $rw_r - sw_s = 2xw_x$.



$r(w_x x_r + w_y y_r) - s(w_x x_s + w_y y_s)$

$r\left(w_x \frac{1}{s} + w_y s\right) - s\left(w_x\left(\frac{-r}{s^2}\right) + w_y r\right)$

$\frac{r}{s} w_x + \cancel{w_y sr} + \frac{r}{s} w_x \cancel{- s w_y r}$

$= 2\frac{r}{s} w_x = 2x w_x$

$x_r = 1/s$

$x_s = \frac{-r}{s^2}$

$y_s = r$

$y_r = s$

b) **(7 points)** Find w_{rs}.

$\frac{\partial}{\partial s}\left(\frac{\partial w_r}{\partial r}\right)$

$w_r = w_x x_r + w_y y_r$

$w_r = w_x \frac{1}{s} + w_y s$

$w_{rs} = \left(w_x \frac{1}{s} + w_y s\right)_s$

$\cancel{w_{rs} = (w_{xx} x_s)\frac{1}{s} + \left(-\frac{1}{s^2} w_x\right) + (w_{yy} y_s}$

$w_{rs} = (w_{xx} x_s + w_{xy} y_s)\frac{1}{s} + \left(\frac{-1}{s^2} w_x\right) + (w_{yx} x_s + w_{yy} y_s)s + w_y$

$w_{rs} = \left(w_{xx}\left(\frac{-r}{s^2}\right) + w_{xy} r\right)\frac{1}{s} - \frac{1}{s^2} w_x + \left(w_{yx}\frac{1}{s} + w_{yy} r\right)s + w_y$

$w_{rs} = \left(-\frac{r}{s^3} w_{xx} + \frac{r}{s} w_{xy} - \frac{w_x}{s^2} + w_{yx} + sr\, w_{yy} + w_y\right.$

$f_x = 2xy + e^y$
$f_y = x^2 + xe^y$

5) (20 points) Consider the function $z = f(x,y) = x^2y + xe^y + 2$, the point $A(1,0)$ in its domain, and the vector $\vec{v} = 4\vec{i} - 3\vec{j}$.

$U = \frac{\vec{V}}{\|V\|}$

a) **(7 points)** Find the derivative at A in the direction of $\vec{v}$.

$\vec{\nabla f} = f_x\vec{i} + f_y\vec{j}$

$\vec{\nabla f} = (2xy + e^y)\vec{i} + (x^2 + xe^y)\vec{j}$

$\Rightarrow \vec{\nabla f}\Big|_{(1,0)} = (2\times 1\times 0 + e^0)\vec{i} + (1 + 1e^0)\vec{j}$

$\nabla f\Big|_{(1,0)} = \vec{i} + 2\vec{j}$

$D_{\vec{u}} f = \vec{\nabla f} \times \vec{u}$

~~$D_u f$ ...~~

$D_{\vec{u}} f = 1\times\frac{4}{5} - \left(2\times\frac{3}{5}\right)$

$D_{\vec{u}} f = \frac{4}{5} - \frac{6}{5} = \frac{-2}{5}$

$\|\nabla f\| = \frac{i}{\sqrt{5}} + \frac{2j}{\sqrt{5}}$

$\frac{4i}{5} - \frac{3j}{5}$

b) **(3 points)** Estimate the change in z if $P(x,y)$ moves away from A a distance of $ds = 0.1$ units in the direction of $\vec{v}$.

$D_{\vec{u}} f \times ds = \frac{-2}{5}\times\frac{1}{10} = \frac{-2}{50} = \frac{-1}{25}$

c) **(4 points)** Find the direction in which the function increases most rapidly at A, and find the derivative of the function in this direction.

$\|\vec{\nabla f}\| = \frac{i}{\sqrt{5}} + \frac{2j}{\sqrt{5}}$ This is the function increases the most

Deriv = ?

-1

d) **(6 points)** Find an equation for the plane that is tangent to the given surface at $B(1,0,3)$.

$\vec{\nabla f} = (2xy + e^y)\vec{i} + (x^2 + xe^y)\vec{j}$

$\nabla f\Big|_{(1,0,3)} = i + 2j$

$1(x-1) + 2(y-0) + 0 = 0$

$x - 1 + 2y = 0$

$\boxed{x + 2y = 1}$

-4

$/5$

6. (14 points) Determine the local maxima, local minima, and saddle points of the function: $f(x,y)=x^3+y^3-9xy+27$.

$f_x = 3x^2 - 9y$ $\qquad f_x = 0 \Rightarrow 3x^2 - 9y = 0 \Rightarrow 9y = 3x^2 \Rightarrow y = $

$f_y = 3y^2 - 9x$ $\qquad f_y = 0 \Rightarrow 3y^2 - 9x = 0 \Rightarrow$

$3\frac{x^4}{9} - 9x = 0 \Rightarrow \frac{x^4}{3} - 9x = 0$

$\Rightarrow x^4 - 27x = 0 \Rightarrow x(x^3-27) = 0 \Rightarrow x = 0$ or $x = \sqrt[3]{27} = 3$

for $x=0 \Rightarrow -9y=0 \Rightarrow \boxed{y=0}$

for $x=3 \Rightarrow 27-9y=0 \Rightarrow -9y=-27 \Rightarrow \boxed{y=3}$

critical point $(0,0)$ and $(3,3)$

$D = f_{xx}f_{yy} - (f_{xy})^2$ $\qquad f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -9$

at point $(0,0)$

$D = -(-9)^2 = -81 < 0$ saddle point

at point $(3,3)$

$D = 18 \times 18 - 81 = 243 > 0$

$f_{xx} > 0$

$f_{yy} > 0$

has a local minim at $(3,3)$

7) (12 points) Find the point on the surface $x^2 + 2y + z - 1 = 0$ that is closest to the origin.

~~Let M(x_M, y_M, z_M)~~

$\cancel{\vec{\nabla f} = \lambda \nabla g}$

$\cancel{D = \sqrt{x^2 + y^2 + z^2}}$

$f(x) = x^2 + y^2 + z^2$

$\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

$\vec{\nabla f} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\vec{\nabla g} = 2x\vec{i} + 2\vec{j} + 1\vec{k}$

$\vec{\nabla f} = \lambda \vec{\nabla g}$

$2x =$

/-7