

NDU

MAT 224

Calculus IV

Exam # 1

Thursday March 30, 2006

Duration: 60 minutes

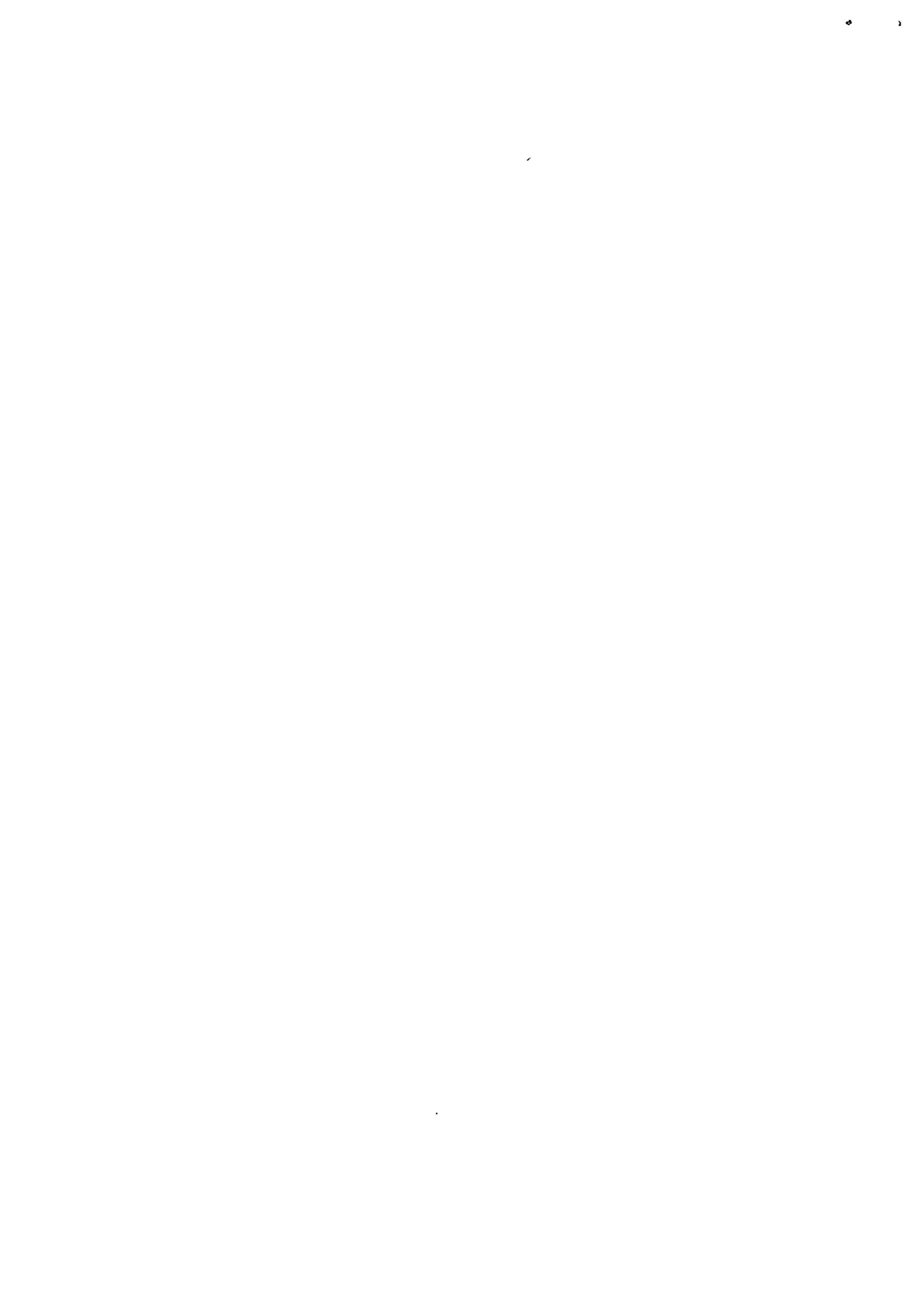
Name: Ammonia Hui Tuohu

Section: _____

Instructor: Dr. Roy

Grade: 93

Problem Number	Points	Score
1	20	
2	15	
3	15	
4	10	
5	10	
6	15	
7	15	
Total	100	



1) (20 points) Let $f(x, y) = \frac{1}{y - x^2}$.

a) (3 points) Find the function's domain.

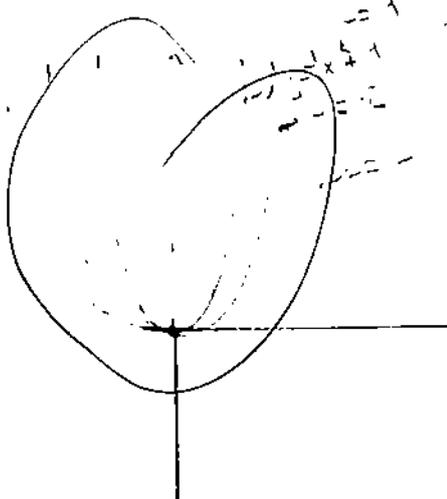
$$y - x^2 \neq 0$$

b) (3 points) Find the function's range.

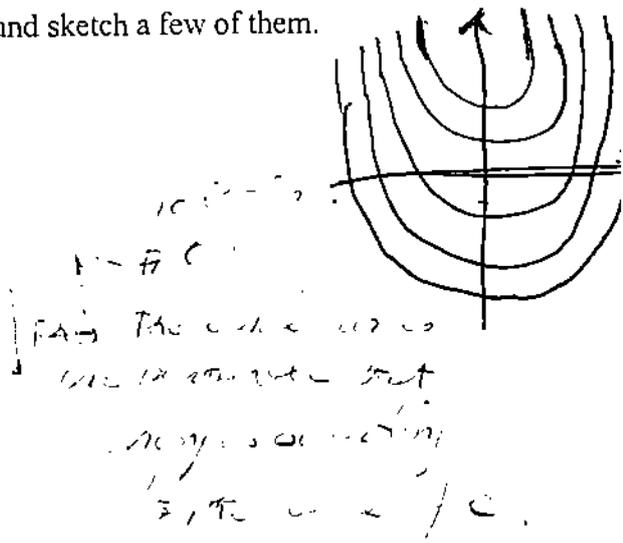
$$y = 902 \cup 30, +\infty$$

18

c) (4 points) Describe the function's level curves, and sketch a few of them.



$$\begin{aligned} c(x, y) &= - \\ -x^2 &= - \\ -x^2 &= 1 \\ -x^2 &= 1 \\ x &= x^2 + 1 \end{aligned}$$



d) (3 points) Find the boundary of the function's domain.

$$y = x^2$$

e) (4 points) Determine if the domain is an open region, a closed region or neither, and explain why.

The domain is an open region because it does not include the boundary points.

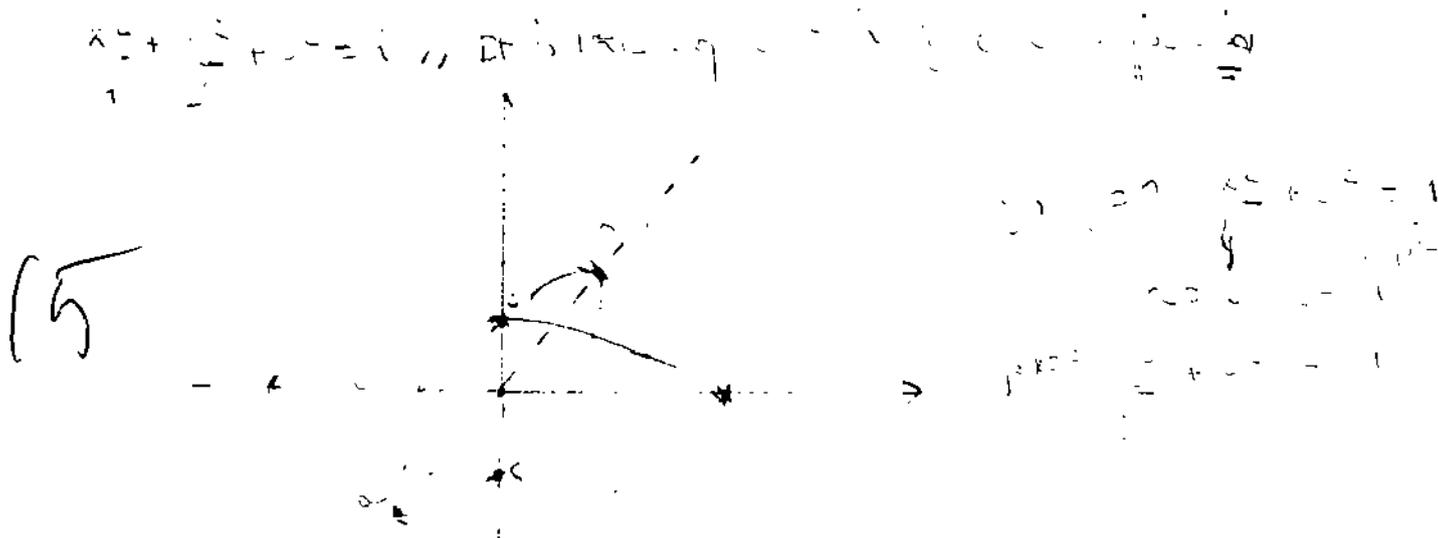
f) (3 points) Decide if the domain is bounded or unbounded.

The domain is unbounded.



2) (15 points) Consider the surface (S): $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$.

a) (3 points) Sketch (S) and identify it. (Details not required)



b) (6 points) Find an equation of the tangent plane (T) to (S) at $P\left(1, 0, \frac{\sqrt{3}}{2}\right)$.

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

$$F(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2 - 1 = 0$$

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = \left(\frac{x}{2}, \frac{2y}{9}, 2z \right)$$

$$\nabla F \Big|_P = \left(\frac{1}{2}, \frac{2 \cdot 0}{9}, 2 \cdot \frac{\sqrt{3}}{2} \right) = \left(\frac{1}{2}, 0, \sqrt{3} \right)$$

$$T: \frac{1}{2}(x-1) + 0(y-0) + \sqrt{3}\left(z - \frac{\sqrt{3}}{2}\right) = 0$$

$$\frac{1}{2}x - \frac{1}{2} + \sqrt{3}z - \frac{3}{2} = 0$$

$$\frac{1}{2}x + \sqrt{3}z - 2 = 0$$

$$x + 2\sqrt{3}z - 4 = 0$$

c) (6 points) Find parametric equations for the line (L) tangent to the curve of intersection of the surface (S) and the plane $z = \frac{\sqrt{3}}{2}x + y$ at $\left(1, 0, \frac{\sqrt{3}}{2}\right)$.

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

$$z = \frac{\sqrt{3}}{2}x + y$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \left(\frac{\sqrt{3}}{2}x + y\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{3}{4}x^2 + \sqrt{3}xy + y^2 = 1$$

$$\frac{7}{4}x^2 + \sqrt{3}xy + \frac{10}{9}y^2 = 1$$

$$\vec{r} = x\vec{i} + y\vec{j} + \left(\frac{\sqrt{3}}{2}x + y\right)\vec{k}$$

$$\vec{r}' = x'\vec{i} + y'\vec{j} + \left(\frac{\sqrt{3}}{2}x' + y'\right)\vec{k}$$

$$\vec{r}' = \vec{i} + \sqrt{3}\vec{k}$$



3) (15 points)

a) (7 points) Find the following limit or show that it does not exist: $f(x) = \frac{x^4 - y^2}{x^4 + y^2}$

Let $y = kx^2$

$$\frac{x^4 - (kx^2)^2}{x^4 + (kx^2)^2} = \frac{x^4 - k^2 x^4}{x^4 + k^2 x^4} = \frac{x^4(1 - k^2)}{x^4(1 + k^2)} = \frac{1 - k^2}{1 + k^2}$$

$x^2 + y^2 \neq 0$
 $x \neq 0$
 $y = kx^2$

b) (8 points) Let $f(x, y) = \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$. If possible, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

$x^2 + y^2 = r^2$
 $x = r \cos \theta$
 $y = r \sin \theta$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{r \rightarrow 0} \ln\left(\frac{3r^2 \cos^2 \theta - r^4 \cos^2 \theta \sin^2 \theta + 3r^2 \sin^2 \theta}{r^2}\right)$$

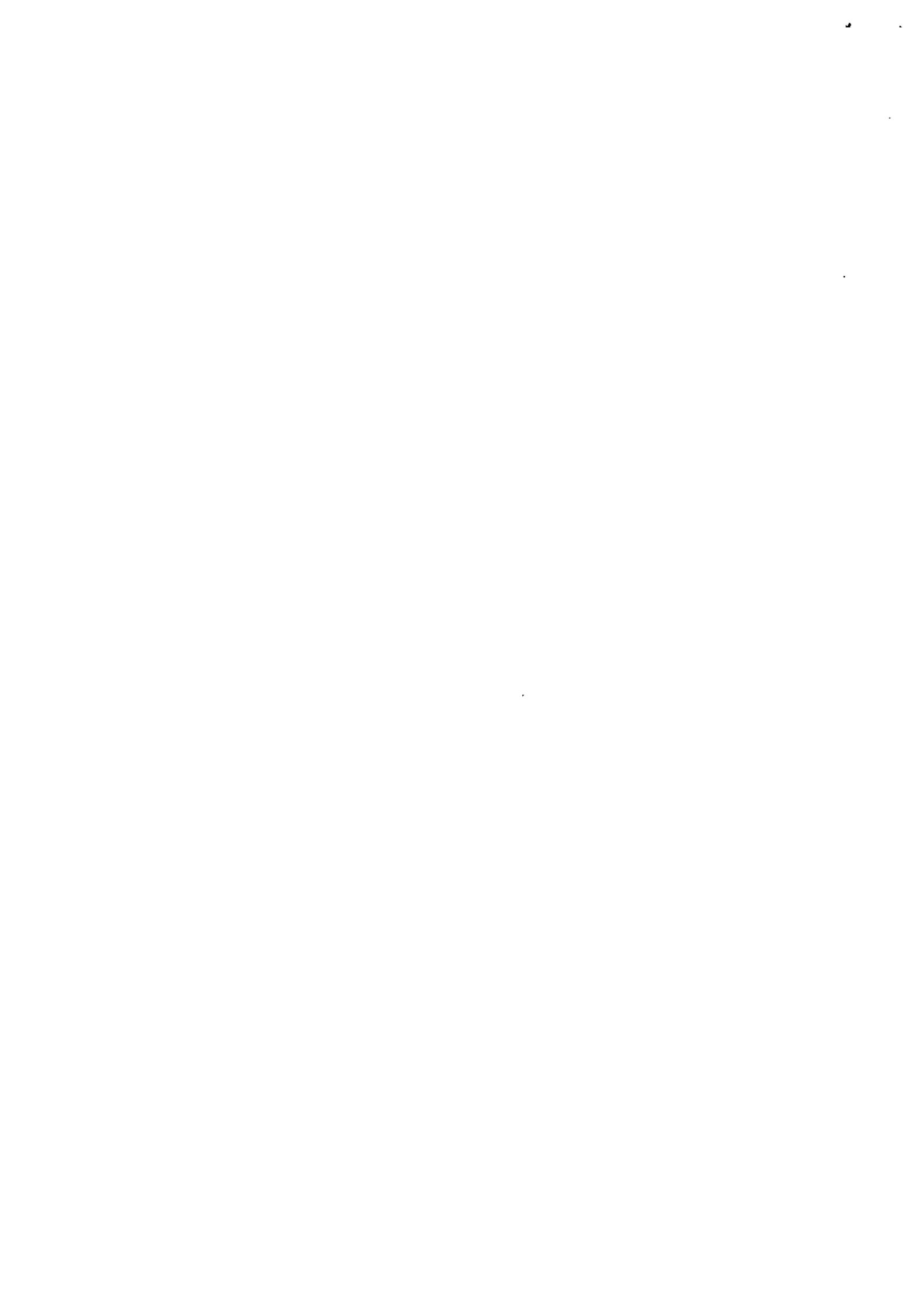
$$= \lim_{r \rightarrow 0} \ln\left(\frac{3r^2 \cos^2 \theta - r^2 \cos^2 \theta \sin^2 \theta + 3r^2 \sin^2 \theta}{r^2}\right)$$

$$= \lim_{r \rightarrow 0} \ln\left(\frac{3r^2(1 - \cos^2 \theta \sin^2 \theta)}{r^2}\right)$$

$$= \lim_{r \rightarrow 0} \ln(3 - \cos^2 \theta \sin^2 \theta)$$

$f(0,0) = \ln 3$

\Rightarrow for f to be continuous at the origin, let $f(0,0) = \ln 3$



- 4) (10 points) Find the value of $\frac{\partial z}{\partial x}$ at the point $P(-3, -1, 1)$ if the equation $xz + y \ln z - z^2 + 4 = 0$ defines z as a function of the two independent variables x and y , and the partial derivatives exist.

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$$xz + y \ln z - z^2 + 4 = 0$$

$$\frac{\partial}{\partial x} (xz + y \ln z - z^2 + 4) = 0$$

$$z + x \frac{\partial z}{\partial x} + \frac{y}{z} \frac{\partial z}{\partial x} - 2z \frac{\partial z}{\partial x} = 0$$

$$z + x \frac{\partial z}{\partial x} + \frac{y}{z} \frac{\partial z}{\partial x} - 2z \frac{\partial z}{\partial x} = 0$$

$$z - z \frac{\partial z}{\partial x} + \frac{y}{z} \frac{\partial z}{\partial x} = 0$$

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at point

$P(-3, -1, 1)$

- 5) (10 points) The derivative of $f(x, y)$ at $P(1, 2)$ in the direction of $\vec{i} + \vec{j}$ is $2\sqrt{2}$, and in the direction of $-2\vec{j}$ is -3 . Find the gradient of f at $P(1, 2)$.

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$$\vec{u} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$$

$$\vec{v} = -2\vec{j}$$

$$\nabla f \cdot \vec{u} = 2\sqrt{2}$$

$$\nabla f \cdot \vec{v} = -3$$

$$\nabla f = \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \right) \cdot \left(\frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j} \right) = 2\sqrt{2}$$

$$\frac{1}{\sqrt{2}} \frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}} \frac{\partial f}{\partial y} = 2\sqrt{2}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4$$

$$\nabla f \cdot \vec{v} = -3$$

$$\left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \right) \cdot (-2\vec{j}) = -3$$

$$-2 \frac{\partial f}{\partial y} = -3 \Rightarrow \frac{\partial f}{\partial y} = \frac{3}{2}$$

$$\frac{\partial f}{\partial x} + \frac{3}{2} = 4 \Rightarrow \frac{\partial f}{\partial x} = \frac{5}{2}$$

$$\nabla f = \frac{5}{2} \vec{i} + \frac{3}{2} \vec{j}$$

$$\nabla f = \frac{5}{2} \vec{i} + \frac{3}{2} \vec{j}$$

6) (15 points) Let $T = g(x, y)$ be the temperature at the point (x, y) on the ellipse $x = 2\sqrt{2} \cos t, y = \sqrt{2} \sin t, 0 \leq t \leq 2\pi$, and suppose that $\frac{\partial T}{\partial x} = y, \frac{\partial T}{\partial y} = x$. Locate and identify the points on the ellipse where the maximum and minimum temperatures occur by examining $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

$x = 2\sqrt{2} \cos t$
 $y = \sqrt{2} \sin t$
 $0 \leq t \leq 2\pi$

$y = \frac{\partial T}{\partial x}$
 $x = \frac{\partial T}{\partial y}$

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$\frac{dT}{dt} = \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt}$

$= y \cdot (-2\sqrt{2} \sin t) + x \cdot (-\sqrt{2} \sin t)$
 $= -2\sqrt{2} y \sin t + \sqrt{2} x \cos t$

~~$\frac{dT}{dt} = -2\sqrt{2} y \sin t + \sqrt{2} x \cos t$~~

$\frac{dT}{dt} = -2\sqrt{2} (\sqrt{2} \sin t) (\sin t) + \sqrt{2} (2\sqrt{2} \cos t) \cos t$
 $\frac{dT}{dt} = -4 \sin^2 t + 4 \cos^2 t$

$\frac{dT}{dt} = 0 \Rightarrow \sin^2 t = \cos^2 t$
 $\sin^2 t - \cos^2 t = 0 \Rightarrow \pm \cos t$
 $\sin t = \cos t \Rightarrow \sin t = -\cos t$

for $t = \pi/4$

$x = 2\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2$
 $y = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1$

$t = 3\pi/4$

$y = 1$

$\sin t = -\pi/4 \Rightarrow$

$x = 2, y = -1$

$\sin t = \cos t \Rightarrow \sin t = \cos t$
 $\sin t = -\cos t \Rightarrow \sin t = -\cos t$

$t = \pi/4$

$\cos t = -\sin t$

$t = 3\pi/4$

$\frac{d^2T}{dt^2} = -8 \sin t \cos t - 8 \cos t \sin t = -16 \sin t \cos t$

for $t = \pi/2$

$x = 0$ if n is odd $y = -\sqrt{2}$
 $x = 0$ if n is even $y = \sqrt{2}$

$\sin t \cos t = 0$

$t = \pi/2$

$t = \pi$

for $t = \pi$
 $x = -2\sqrt{2}$ (if n is even)
 $x = 2\sqrt{2}$ (if n is odd)

conclusion

The points are $(2, 1)$ } ~~local maximum~~
 $(2, -1)$ } ~~local maximum~~
 $(0, -\sqrt{2})$ } minimum
 $(0, \sqrt{2})$ }
 $(2\sqrt{2}, 0)$ } maximum.
 $(-2\sqrt{2}, 0)$ }

$$\frac{dT}{dt} = 0 \Rightarrow \sin t \pm \cos t$$

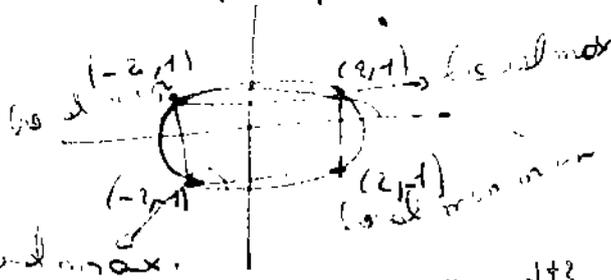
$$\Rightarrow t = \pi/4 ; -\pi/4, \frac{3\pi}{4}, \frac{5\pi}{4}$$

$$t = \pi/4, \quad x = 2$$

$$y = 1$$

$$t = -\pi/4 ;$$

$$\frac{v=2}{y=-1}$$



connected by

$$\frac{d^2T}{dt^2} = -\sin t \pm \cos t = -\cos t \pm \sin t \quad \text{when } t = \pi/4 \quad \frac{d^2T}{dt^2} < 0$$



- 7) (15 points) Determine the local maxima, local minima, and saddle points of the function: $f(x, y) = x^3 - 6xy + y^2$.

$$\begin{aligned} f_x &= 3x^2 - 6y = 0 \\ f_y &= 2y - 6x = 0 \end{aligned}$$

$$\begin{aligned} 2y - 6x &= 0 \\ 2y &= 6x \\ y &= 3x \end{aligned}$$

$$\text{If } x=0 \text{ then } y=0$$

$$\text{If } x=6 \text{ then } y=18$$

$$3x^2 - 6(3x) = 0$$

$$3x^2 - 18x = 0$$

$$x(3x - 18) = 0$$

$$x=0 \text{ or } 3x - 18 = 0$$

$$3x = 18$$

$$\boxed{x = 6}$$

$$y = 3x$$

$$y = 18$$

$$f_{xy} = -6$$

$$f_x(0,0) = 3x^2 = 0$$

$$f_y(0,0) = 2y = 0$$

$$f_{xy}(0,0) = -6$$

$$f_{xx} f_{yy} - (f_{xy})^2 = -36 < 0$$

f has a saddle point at $(0, 0)$.
 $\boxed{f(0,0) = 0}$

$$f(6, 18)$$

$$f_{xx} = 36$$

$$f_{yy} = 2$$

$$f_{xy} = -6$$

$$f_{xx} f_{yy} - (f_{xy})^2 = 72 - 36 > 0$$

$$\text{and } f_{xx} > 0$$

So f has a local minimum at $(6, 18)$

$$f(6, 18) = 6^3 - 6(6)(18) + 18^2$$

$$\boxed{f(6, 18) = -108}$$

