Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 224
CALCULUS IV
Exam #1
Thursday April 14, 2005
Duration: 60 minutes

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Section: ________________

Instructor: Dr. Rached

Grade: 93

Problem Number	Points	Score
1	10	
2	5	
3	30	
4	16	
5	9	
6	6	
7	8	
8	16	
Total	100	

5

1. (10 points)

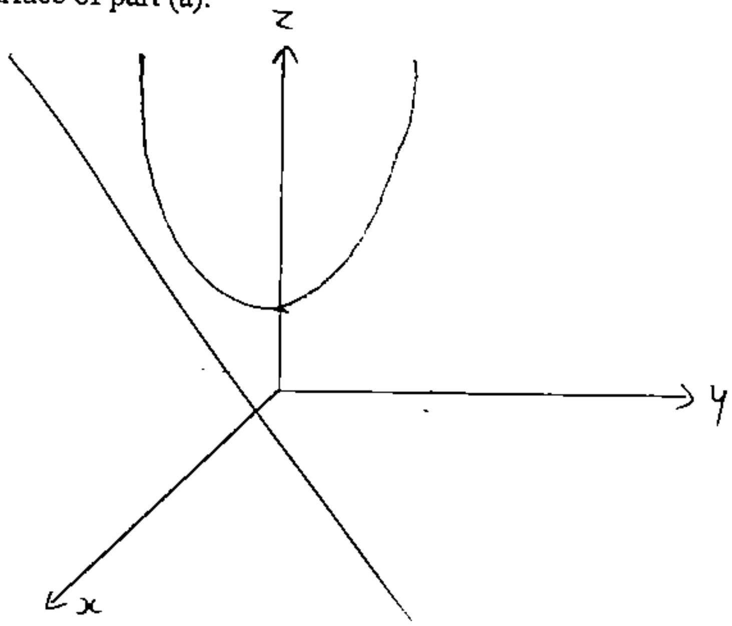
a) Find an equation for the level surface of the function $f(x,y,z) = y^2 + \frac{z^2}{4}$ through the point (0, 1, 2).

$f(x,y,z) = y^2 + \frac{z^2}{4}$ level surface at the point $(0,1,2)$

$f(x,y,z) = 1 + \frac{4}{4} = 2.$

$\Rightarrow y^2 + \frac{z^2}{4} = 2.$

b) Sketch the level surface of part (a).



5

2. (5 points) Show that a moving particle will move in a straight line if the normal component of its acceleration is zero.

$a = a_T T + a_N N$. The normal component of acceleration is a_N

~~$a_T = \frac{d}{dt}|v|$ for a_T to be zero, $|v|$ must be constant~~

$a_N = K|v|^2$ for a_N to be zero ~~should~~ the curvature K should be equal to zero ~~and if $K=0$ ⇒ the curve is a straight~~ because the particle is moving which means $|v| \neq 0$ and if $K = 0 \Rightarrow$ the curve is a straight line.

$(e^t\sqrt{2})^3$
e^{3t}.

$e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t$

$(2e^t \cos t) e^t\sqrt{2} - e^{2t}\sqrt{2}\sin t - e^{2t}\sqrt{2}\cos t$

$2\sqrt{2}e^{2t}\cos t - e^{2t}\sqrt{2}\sin t - e^{2t}\sqrt{2}$

$\sqrt{2}e^{2t}\cos t - e^{2t}\sqrt{2}\sin t$ $\quad (e^t\cos t - e^t \sin t)(\cos t - \sin t)$

$\frac{\sqrt{2}e^{2t}(\cos t - \sin t)}{2e^{2t}} = \cos t - e^t\cos^2 t + e^t \sin^2 t$

$\frac{e^t 0}{2e^t}$ $\frac{1}{2}$ $\quad 2e^t$

$\left(\frac{e^t\cos t - e^t \sin t}{e^t\sqrt{2}}\right)$.

$(-\sin t - \cos t)e^t \sin t + e^t \cos t$

$-e^t\sin^2 t - e^t \cos^2 t$

$\frac{-e^t 0}{2e^t} = -\frac{1}{2}$

$e^t \cos t - e^t \sin t$

$u'v + uv'$

$e^t\sqrt{2}$

$e^t\sqrt{2} +$

$\frac{e^t \cos t - e^t\sin t - (e^t\sin t + e^t \cos t)}{e^t\cos t - e^t \sin t - e^t \sin t - e^t\cos t}$

$\frac{(-2e^t \sin t)e^t\sqrt{2} - (e^t\cos t - e^t\sin t)e^t\sqrt{2}}{2e^{2t}}$.

$-\cancel{2e^{2t}} - 2\sqrt{2}e^{2t}\sin t - e^{2t}\cos t\sqrt{2} + e^{2t}\sqrt{2}\sin t$

$2\sqrt{2} - \sqrt{2}$.

$\sqrt{2}(2-1) = 1\sqrt{2}$.

$\frac{-\sqrt{2}e^{2t}\sin t - e^{2t}\sqrt{2}\cos t}{2e^{2t}}$

$\frac{-\sqrt{2}\cancel{e^{2t}}(\sin t + \cos t)}{2\cancel{e^{2t}}}$

$\frac{-\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$ $\quad -\left(\frac{\sin t + \cos t}{\sqrt{2}}\right)$

$(e^t\sqrt{2})^3$

e^{3t}.

$e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t$

$(2e^t\cos t)e^t\sqrt{2} - e^{2t}\sqrt{2}\sin t - e^{2t}\sqrt{2}\ldots$

$2\sqrt{2}e^{2t}\cos t - e^{2t}\sqrt{2}\sin t - e^{2t}\sqrt{2}$

$\sqrt{2}e^{2t}\cos t - e^{2t}\sqrt{2}\sin t \qquad (e^t\cos t - e^t\sin t)(\cos t - \sin t)$

$\dfrac{\cancel{\sqrt{2}}\cancel{e^{2t}}(\cos t - \sin t)}{2\cancel{e^{2t}}} = \cos t - e^t\cos^2 t + e^t\sin^2 t$

$\dfrac{e^t 0}{2\cancel{e^t}} \quad \dfrac{1}{2} \qquad 2e^t$

$\left(\dfrac{e^t\cos t - e^t\sin t}{e^t\sqrt{2}}\right)$

$(-\sin t - \cos t)e^t\sin t + e^t\cos\ldots$

$-e^t\sin^2 t - e^t\cos^2 t$

$e^t\cos t - e^t\sin t$

$u'v + uv'$

$e^t\sqrt{2}$

$\dfrac{-e^t 0}{2e^t} = -\dfrac{1}{2}$

$e^t\sqrt{2} + 0$

$e^t\underline{\cos t - e^t\sin t - (e^t\sin t + e^t\cos t)}$

$e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t$

$\dfrac{(-2e^t\sin t)e^t\sqrt{2} - (e^t\cos t - e^t\sin t)e^t\sqrt{2}}{2e^{2t}}$.

$-\cancel{2e^{2t}} - 2\sqrt{2}e^{2t}\sin t - e^{2t}\cos t\sqrt{2} + e^{2t}\sqrt{2}\sin t$

$\dfrac{-\sqrt{2}e^{2t}\sin t - e^{2t}\sqrt{2}\cos t}{2e^{2t}}$

$2\sqrt{2} - \sqrt{2}$.

$\sqrt{2}(2-1) = 1\sqrt{2}$.

$\dfrac{-\sqrt{2}\cancel{e^{2t}}(\sin t + \cos t)}{2\cancel{e^{2t}}}$.

$\dfrac{-\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} \quad -\left(\dfrac{\sin t + \cos t}{\sqrt{2}}\right)$

30

3. **(30 points)** Given the space curve $\vec{r}(t)=(e^t\cos t)i+(e^t\sin t)j+2k$, find

a) **(20 points)** T, N, B, κ, and τ at $t=0$.

$r(t)=(e^t\cos t)\vec{i}+(e^t\sin t)\vec{j}+2\vec{k}$

$v(t)=(e^t\cos t-e^t\sin t)\vec{i}+(e^t\sin t+e^t\cos t)\vec{j}$

$|v(t)|=\sqrt{(e^t\cos t-e^t\sin t)^2+(e^t\sin t+e^t\cos t)^2}$

$=\sqrt{e^{2t}\cos^2 t+e^{2t}\sin^2 t-\cancel{e^{2t}\cos t\sin t}+e^{2t}\sin^2 t+e^{2t}\cos^2 t+\cancel{e^{2t}\cos t\sin t}}$

$=\sqrt{e^{2t}(\cos^2 t+\sin^2 t)+e^{2t}(\sin^2 t+\cos^2 t)}=\sqrt{e^{2t}+e^{2t}}$

$=\sqrt{2e^{2t}}=e^t\sqrt{2}$

$T=\left(\frac{e^t\cos t-e^t\sin t}{e^t\sqrt{2}}\right)\vec{i}+\left(\frac{e^t\sin t+e^t\cos t}{e^t\sqrt{2}}\right)\vec{j}$ $\qquad T=\frac{v}{|v|}$

$T(0)=\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}$

$\left(\frac{(e^t\cos t-e^t\sin t-e^t\sin t+e^t\cos t)e^t\sqrt{2}-(e^t\cos t-e^t\sin t)e^t\sqrt{2}}{2e^{2t}}\right)\vec{i}$

$\left(\frac{(-2e^t\sin t)e^t\sqrt{2}-e^{2t}\sqrt{2}\cos t+e^{2t}\sqrt{2}\sin t}{2e^{2t}}\right)\vec{i}=\left(\frac{-2\sqrt{2}e^{2t}\sin t-e^{2t}\sqrt{2}\cos t+e^{2t}\sqrt{2}\sin t}{2e^{2t}}\right)$

$=-\frac{\sqrt{2}e^{2t}\sin t-e^{2t}\sqrt{2}\cos t}{2e^{2t}}\vec{i}=\frac{-e^{2t}\sqrt{2}(\sin t+\cos t)}{2e^{2t}}\vec{i}$

$=-\frac{\sin t-\cos t}{\sqrt{2}}\vec{i}.$

$\left(\frac{(e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t)e^t\sqrt{2}-(e^t\sin t+e^t\cos t)e^t\sqrt{2}}{2e^{2t}}\right)\vec{j}$

$=\frac{\cos t-\sin t}{\sqrt{2}}\vec{j}$ $\qquad N=\frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$

$N=\left(-\frac{\sin t-\cos t}{\sqrt{2}}\right)\vec{i}+\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\vec{j}\Rightarrow N(0)=-\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}$

$B=T\times N$ $\quad\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{e^t\cos t-e^t\sin t}{e^t\sqrt{2}} & \frac{e^t\sin t+e^t\cos t}{e^t\sqrt{2}} & 0\\ \frac{-\sin t-\cos t}{\sqrt{2}} & \frac{\cos t-\sin t}{\sqrt{2}} & 0\end{vmatrix}$

$B=\vec{k}\left[\frac{(e^t\cos t-e^t\sin t)(\cos t-\sin t)}{2e^t}-\frac{(e^t\sin t+e^t\cos t)(-\sin t-\cos t)}{2e^t}\right]$

$(e^t \cos t - e^t \sin t)$

$e^t \cos t - e^t \sin t - (e^t \sin t + e^t \cos t)$

$-2e^t \sin t$

$e^t \sin t + e^t \cos t$

$e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t$

$2e^t \cos t$

$2e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t$

$(e^t \cos t - e^t \sin t)$.

$-(-2e^{2t} \sin^2 t - 2e^{2t} \sin t \cos t)$

$2e^{2t} \cos^2 t + 2e^{2t} \sin^2 t$

$2e^{2t} \cancel{}$

$-2e^t \sin t - 2e^t \cos t$

$-2e^t (\sin t + \cos t$

oscillating

Normal $\quad 2e^t \cos t - 2e^t \sin$

rectifying

$B = \vec{k}$

$a(t) = (-2e^t \sin t)\,\hat{i} + (2e^t \cos t)\,\vec{j}$. 3/7

$K = \dfrac{|v \times a|}{|v|^3}$ ~~$v \times a = \quad e^t\cos t -$~~

$v \times a = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ e^t\cos t - e^t\sin t & (e^t\sin t + e^t\cos t) & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \end{vmatrix}$ ~~$= \vec{k}\,(\ldots)$~~ $= 2e^{2t}\vec{K}$ $\quad |v \times a| = \sqrt{4e^{4t}} = 2e^{2t}$

$|v|^3 = e^{3t}(2\sqrt{2})$ $\qquad K = \dfrac{2e^{2t}}{2\sqrt{2}e^{3t}} = \dfrac{1}{\sqrt{2}e^t} \Rightarrow K(0) = \dfrac{1}{\sqrt{2}}$

$\tau = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$ $\qquad \tau = \begin{vmatrix} e^t\cos t - e^t\sin t & e^t\sin t + e^t\cos t & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \\ -2e^t\sin t - 2e^t\cos t & 2e^t\cos t - 2e^t\sin t & 0 \end{vmatrix} = 0$.

b) **(5 points)** the equation of the osculating plane at $t = 0$.

osculating plane. passes by $r(0)$ and orthogonal to $B(0)$

$r(0) = \hat{i} + 2\vec{k}$ $\qquad B = \vec{k}$.

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ $\quad$ A, B, C coeficients of $B(0)$

$A = B = 0 \qquad 1(z - 2) = 0$

$z - 2 = 0$

$z = 2$.

c) **(5 points)** the length of the portion of the curve between $t = 0$ and $t = 1$.

$L = \int_0^1 |v(t)| = \int_0^1 e^t\sqrt{2} = \sqrt{2}\int_0^1 e^t$.

$= \sqrt{2}\,[e^t]_0^1$

$= \sqrt{2}\,(e - 1)$

$= \sqrt{2}e - \sqrt{2}$.

$\cancel{x}$

$9 - x^2 - y^2 \geq 0.$

$9 - x^2 - y^2$

$9 \geq x^2 + y^2$

$\cancel{x^2 + y^2} =$

$-x^2 - y^2 = -9 + c^2$

$x^2 + y^2 = 9 - c^2$

14.5

4. (16 points) Let $f(x,y)=\sqrt{9-x^2-y^2}$.

a) **(2 points)** Find the function's domain.

~~Domain~~ $9 \geq x^2+y^2$.

All (x,y) respecting $9 \geq x^2+y^2$.

2

b) **(3 points)** Find the fucntion's range.

Range $[0,3]$.

2.5

c) **(4 points)** Describe the function's level curves and sketch few of them.

$\sqrt{9-x^2-y^2}=C$ $\quad 9-x^2-y^2=c^2$

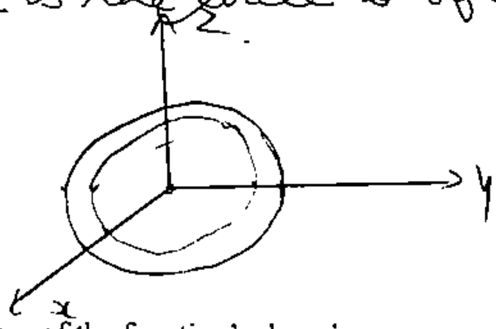
$x^2+y^2=9-c^2$.

if $c=3$ the level curve is the origin $x^2+y^2=0$

~~if $c\neq 3$~~ if $c=1$, $x^2+y^2=8$ $\quad$ if $c=2$ $\quad x^2+y^2=5$.

$c\neq 3$ level curve is the circle of center $(0,0)$ $R=\sqrt{9-c^2}$.

4



d) **(2 points)** Find the boundary of the function's domain.

2 the boundary is the circle $x^2+y^2=9$.

e) **(3 points)** Determine if the domain is an open region, a closed region, or neither.

2 the domain is closed why ?

f) **(2 points)** Decide if the domain is bounded or unbounded.

2 The domain is bounded by the circle $x^2+y^2=9$.

$r(\cos\theta \sin\theta) \cdot \alpha^3 (r^2\cos^2\theta \; r^2\sin^2\theta)$

$\alpha^4 \, r^3\cos^3\theta \sin\theta + r^3\cos\theta$

8.5

5. (9 points) Define $f(0,0)$ in a way that extends $f(x,y)=xy\dfrac{x^2-y^2}{x^2+y^2}$ to be continuous at the origin.

we switch to polar coordinates

$$\lim_{r\to 0} r\cos\theta\, r\sin\theta \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \cancel{\frac{r^2(\cos\theta\sin\theta)(r^2(\cos^2\theta-\sin^2\theta))}{r^2}}$$

$$\lim_{r\to 0} \frac{(r^2\cos\theta\sin\theta)(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta\sin\theta - r^3\cos\theta\sin^3\theta}{r^2}$$

$$\lim_{r\to 0} r^3\frac{(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)}{r^2} = \lim_{r\to 0} r^2(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$$

$= 0.$

$f(0,0)=0$

6

6. (6 points) Find the limit of $f(x,y)=\dfrac{x^2}{x^2-y}$ as $(x,y)\to(0,0)$ or show that the limit does not exist.

$f(x,y) = \dfrac{x^2}{x^2-y}$

we consider different paths

x axis $\Rightarrow y=0$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2} = 1$

y axis $\Rightarrow x=0$ $\lim\limits_{(x,y)\to(0,0)} = \dfrac{0}{-y} = 0$

$\Big\}\ 1 \neq 0$

By the 2 paths of non existence of a limit $\Rightarrow$ the limit doesn't exist because it has different limits at different paths

$1 - x - y^2 - \sin xy$

$f_x = -1$

$xy = u'v + uv'$

$= y + x(0) = y.$

$xy = u'v + uv'$

$0(y) + x(1) = x.$

$-1 - y\cos(xy)$

$-1 - 1\cos(0 \times 1)$

$-1 - 1 \times 1$

$-1 - 1 = (-2) = +2.$

$-2y - x\cos(xy)$

$-2(1) - 0\cos(0 \times 1)$ $\quad \frac{3}{-2} = -1.$

$-2 - 0 \times 1 =$

-2

8

7. (8 points) Assuming that the equation $1-x-y^2-\sin xy=0$ define y as a differentiable function of x, find the value of $\frac{dy}{dx}$ at the point $p(0,1)$.

$$\frac{\partial y}{\partial x} = \frac{-F_x}{F_y} \qquad 1-x-y^2-\sin xy = 0$$

$$F_x = -1 - y\cos(xy)$$
$$F_y = -2y - x\cos(xy)$$
$$\left.\frac{\partial y}{\partial x}\right|_{(0,1)} = -\frac{[-1-y\cos(xy)]}{-2y-x\cos(xy)} = -1.$$

16

8. (16 points) If $f(u,v,w)$ is differentiable and $u=x-y$, $v=y-z$, and $w=z-x$.

a) **(10 points)** Show that $f_x+f_y+f_z=0$.

$$f_x = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$f_x = f_u(1) + f_v(0) + f_w(1)$$
$$f_x = f_u + \cancel{\;} f_w(-1) = f_u - f_w$$

$$f_y = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = -f_u + f_v(1) + f_w(0)$$
$$= -f_u + f_v$$

10

$$f_z = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = f_u(0) - f_v + f_w$$
$$= -f_v + f_w$$

$$f_x + f_y + f_z = f_u \cancel{\;} - f_u + f_v - f_v + f_w$$
$$f_x + f_y + f_z = 0$$

b) **(6 points)** Express f_{xx} in terms of f_{uu}, f_{uw}, and f_{ww}.

$$f_{xx} = \frac{\partial}{\partial x}[f_u] - \frac{\partial}{\partial x}[f_w].$$

$$= \left[\frac{f_u}{\partial u}\frac{\partial u}{\partial x} + \frac{f_u}{\partial v}\frac{\partial v}{\partial x} + \frac{f_u}{\partial w}\frac{\partial w}{\partial x}\right] - \left[\frac{f_w}{\partial u}\frac{\partial u}{\partial x} + \frac{f_w}{\partial v}\frac{\partial v}{\partial x} + \frac{f_w}{\partial w}\frac{\partial w}{\partial x}\right]$$

6

$$f_{xx} = f_{uu}(1) + f_{uv}(0) + f_{uw}(-1) - \left[f_{wu}(1) + f_{wv}(0) + f_{ww}(-1)\right]$$

$$f_{xx} = f_{uu} - f_{uw} - f_{wu} + f_{ww}.$$

$$f_{xx} = f_{uu} - f_{uw} - f_{wu} + f_{ww}.$$