1) **(20 points)** Let $f(x,y)=\dfrac{1}{y-x^2}$.

a) **(3 points)** Find the function's domain.

$y-x^2 \neq 0 \quad y \neq x^2$

$\Rightarrow$ function domi $\{(x,y)\in\mathbb{R}^2 \,/\, y \neq x^2\}$

~~without~~ ~~the~~ ~~boundaries~~ of the parabola $y=x^2$.

b) **(3 points)** Find the function's range.

$y-x^2 \neq 0 \; \Rightarrow \; \frac{1}{y-x^2} \neq 0 \; \Rightarrow$ the function range

$\{(x,y)\ldots\}$ $f(x,y) \neq 0$.

-2

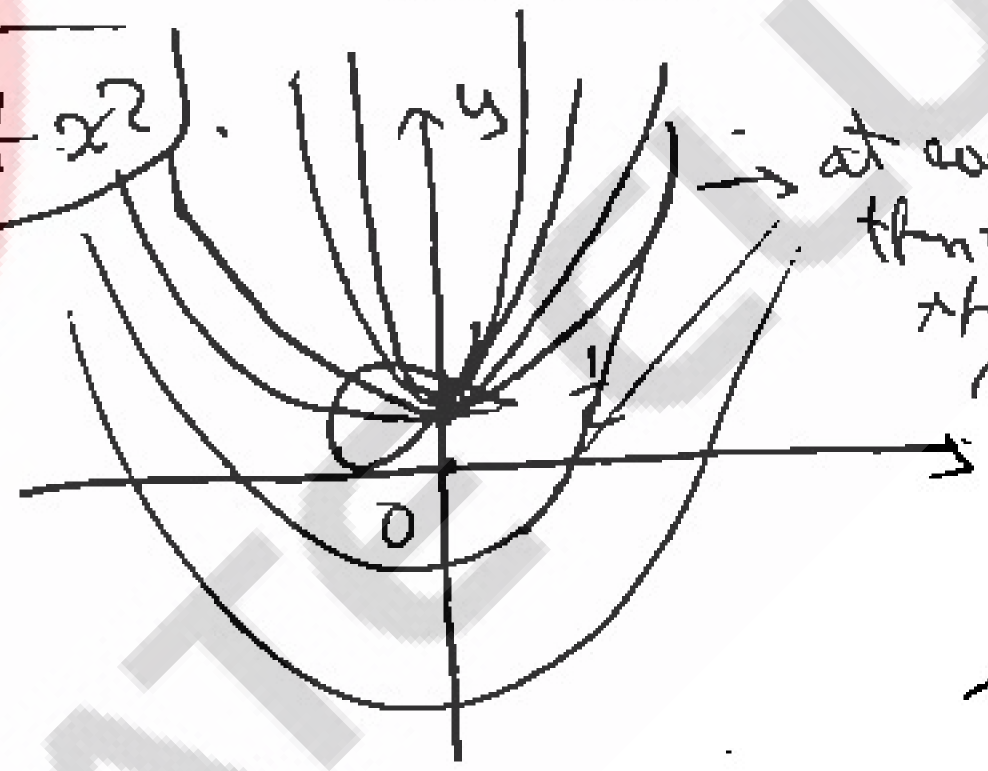
c) **(4 points)** Describe the function's level curves, and sketch a few of them.

let $f(x,y) = c \;/\; c \in \mathbb{R}_f$ ~~$\in \mathbb{R}^2 - \{0\}$~~

$\frac{1}{y-x^2} = c \;\Rightarrow\; \boxed{y = \frac{1}{c} + x^2}$

$c \neq 0$.

$\in$ range.



$\rightarrow$ at each of them the same
the same z
x

-2

d) **(3 points)** Find the boundary of the function's domain.

the boundary is the parabola $y=x^2$.

e) **(4 points)** Determine if the domain is an open region, a closed region or neither, and explain why.

it's an open region because all the points are interior points.

it's not a closed region because it contains not it's boundary.

f) **(3 points)** Decide if the domain is bounded or unbounded.

It's unbounded. because we can't scatch a circle over the parabola because it goes to infinity.

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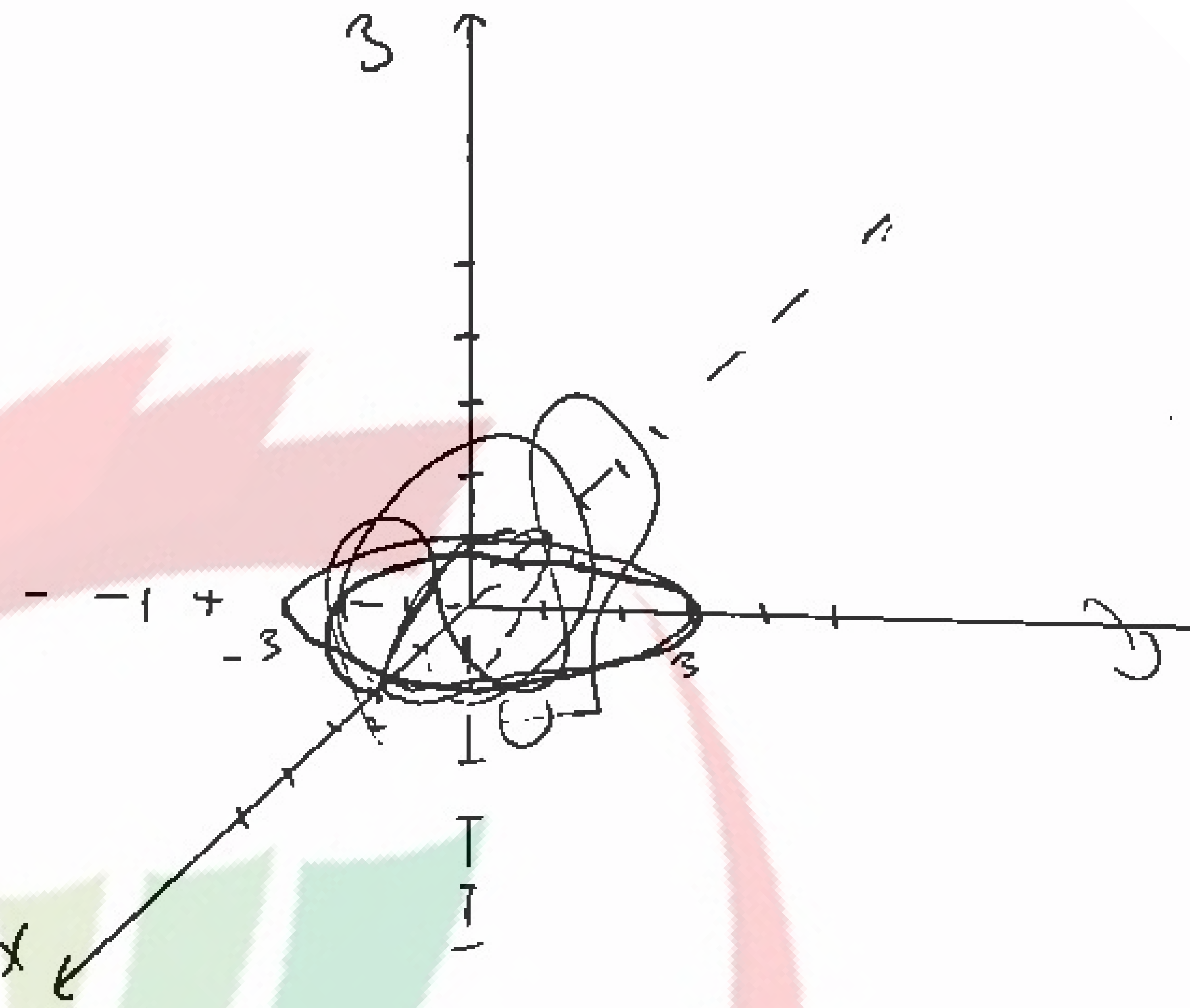
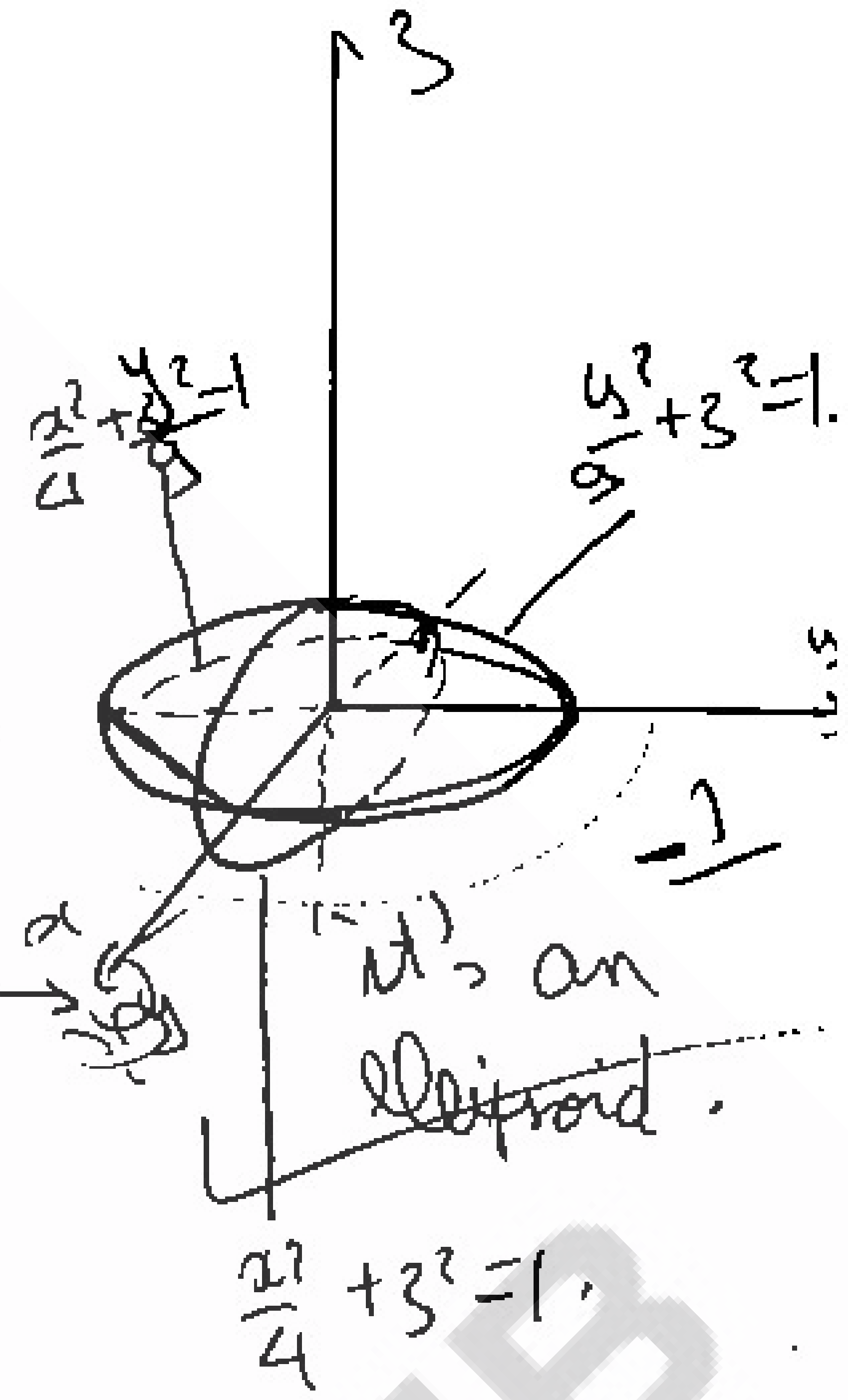
2) (15 points) Consider the surface (S): $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$.

a) **(3 points)** Sketch (S) and identify it. (Details not required)





b) **(6 points)** Find an equation of the tangent plane (T) to (S) at $P\left(1,0,\frac{\sqrt{3}}{2}\right)$.

let's first find the gradient. $\vec{\nabla}_S=\frac{ds}{dx}\vec{i}+\frac{ds}{dy}\vec{j}+\frac{ds}{dz}\vec{k}$

$\vec{\nabla}_{(S)}=\frac{x}{2}\vec{i}+\frac{2y}{9}\vec{j}+2z\vec{k}$. $\quad \vec{\nabla}_S\big|_{1,0,\frac{\sqrt{3}}{2}}=\frac{1}{2}\vec{i}+\sqrt{3}\vec{k}$

$\vec{\nabla}_S$ is $\perp$ ~~to~~ to (S) $\Rightarrow$ it's the normal from the tangent plane (T)

$\Rightarrow$ the eq of (T) is: $\frac{1}{2}x+0y+\sqrt{3}z+d=0$. $P\in(T)$

$\Rightarrow \frac{1}{2}+\frac{3}{2}+d=0 \Rightarrow d=-2$. $\Rightarrow (T)$ is: $\boxed{\frac{1}{2}x+\sqrt{3}z-2=0}$

c) **(6 points)** Find parametric equations for the line (L) tangent to the curve of intersection of the surface (S) and the plane $z=\frac{\sqrt{3}}{2}x+y$ at $\left(1,0,\frac{\sqrt{3}}{2}\right)$.

$\vec{\nabla}_S$ at $1,0,\frac{\sqrt{3}}{2}$ $=\frac{1}{2}\vec{i}+\sqrt{3}\vec{k}$.

$\vec{\nabla}_{(Z)}=\frac{\sqrt{3}}{2}\vec{i}+\vec{j}-\vec{k}$. $\quad \vec{\nabla}_Z\big|_{1,0,\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{2}\vec{i}+\vec{j}-\vec{k}$

the line (L) is tangent to the $\perp$ to (S) and to the perpend to (Z)

$\Rightarrow (L)$ is $\parallel$ to $\vec{\nabla}_S\wedge\vec{\nabla}_{(Z)}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{1}{2}&0&\sqrt{3}\\ \frac{\sqrt{3}}{2}&1&-1\end{vmatrix}=-\sqrt{3}\vec{i}+\vec{j}\left(\frac{1}{2}+\frac{3}{2}\right)+\frac{1}{2}\vec{k}$

P_0P_x is $\parallel$ to (L).

$\Rightarrow (L)$: $\begin{cases}x-1=-\sqrt{3}t\\ y=2t\\ z-\frac{\sqrt{3}}{2}=\frac{1}{2}t\end{cases} \Rightarrow (L)$: $\begin{cases}x=-\sqrt{3}t+1\\ y=2t\\ z=\frac{t}{2}+\frac{\sqrt{3}}{2}\end{cases}$ $/ t\in\mathbb{R}$.

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3) (15 points)

a) **(7 points)** Find the following limit or show that it does not exist: $f(x)=\dfrac{x^4-y^2}{x^4+y^2}$

$\lim_{(x,y)\to(0,0)} f(x) = \frac{0-0}{0+0}$ let's use the two path theorem.

let $y = mx^2 \Rightarrow \lim_{(x,y)\to 0,0} f(x) = \lim_{x\to 0 \text{ and } y=mx^2} \frac{x^4-y^2}{x^4+y^2} = \lim_{x\to 0} \frac{x^4-m^2x^4}{x^4+m^2x^4} = \frac{1-m^2}{1+m^2}$.

the $\lim_{(x,y)\to 0} f(x)$ depends on $m \Rightarrow$ the lim at $(0,0)$ doesn't exist.

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b) **(8 points)** Let $f(x,y)=\ln\left(\dfrac{3x^2-x^2y^2+3y^2}{x^2+y^2}\right)$. If possible, define $f(0,0)$ in a way that extends f to be continuous at the origin.

f is continuous at the origine $\Rightarrow f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$.

$\lim_{(x,y)\to(0,0)} \ln\left(\frac{3x^2-x^2y^2+3y^2}{x^2+y^2}\right)$ let $x = r\cos\alpha$, $y = r\sin\alpha$ $\Rightarrow \lim_{r\to 0} \ln\left(\frac{3r^2-r^4\cos^2\alpha\sin^2\alpha}{r^2(\cos^2\alpha+\sin^2\alpha)}\right)$.

$\Rightarrow = \lim_{r\to 0} \ln\left(\frac{3-r^2\cos^2\alpha\sin^2\alpha}{1}\right)$

$-1<\cos\alpha<1$.
$-1<\sin\alpha<1$.
$\Rightarrow 0<\cos^2\alpha<1$.
$0<\sin^2\alpha<1 \Rightarrow 0<\cos^2\alpha\sin^2\alpha<1$.

$\Rightarrow \lim_{r\to 0} \ln(3-r^2\cos^2\alpha\sin^2\alpha) = \ln 3$ because $\left(\lim_{r\to 0} r^2\cos^2\alpha\sin^2\alpha = 0\right)$

$\Rightarrow f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y) = \ln 3$.

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$f(0,0) = \ln 3$.

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4) **(10 points)** Find the value of $\frac{\partial z}{\partial x}$ at the point $P(-3,-1,1)$ if the equation $xz + y\ln z - z^2 + 4 = 0$ defines z as a function of the two independent variables x and y, and the partial derivatives exist.

Partial derivative exist. (independent variables x).

$$\Rightarrow \frac{\partial z}{\partial x}\Big|_{(-3,-1,1)} = -\frac{F_x}{F_z}\Big|_{(-3,-1,1)} = \frac{-z}{x+\frac{y}{z}-2z}\Big|_{-3,-1,1} = \frac{-1}{-3-1-2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{\cancel{2}6}$$

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5) **(10 points)** The derivative of $f(x,y)$ at $P(1,2)$ in the direction of $\vec{i}+\vec{j}$ is $2\sqrt{2}$, and in the direction of $-2\vec{j}$ is -3. Find the gradient of f at $P(1,2)$.

The derivative $D_{\vec u}f = \vec\nabla f \cdot \vec u$.

let $\vec\nabla f = x\vec i + y\vec j$

Derivative in the direction of $\vec i + \vec j$: $\vec u = \frac{1}{\sqrt2}\vec i + \frac{1}{\sqrt2}\vec j = 2\sqrt2$.

$\vec\nabla f \cdot \vec u \Rightarrow x \times \frac{1}{\sqrt2} + y \times \frac{1}{\sqrt2} = 2\sqrt2 \Rightarrow x + y = 4$.

Derivative the direction of $-2\vec i$ $\vec v = -\vec j = -3$.

$\Rightarrow 0 \times x - y = -3 \Rightarrow y = 3$

$\begin{cases} x+y=4 \\ y=3 \end{cases} \Rightarrow \begin{matrix} x=1 \\ y=3 \end{matrix}$

$\Rightarrow \vec\nabla f\big|_{1,2} = \vec i + 3\vec j$.

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6) (15 points) Let $T = g(x,y)$ be the temperature at the point (x,y) on the ellipse $x = 2\sqrt{2}\ \cos t$, $y = \sqrt{2}\ \sin t$, $0 \le t \le 2\pi$, and suppose that $\frac{\partial T}{\partial x} = y$, $\frac{\partial T}{\partial y} = x$.

Locate and identify the points on the ellipse where the maximum and minimum temperatures occur by examining $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt}.$$

$$= y \times -2\sqrt{2}\sin t + x\cdot\sqrt{2}\cos t. \quad / \ y = \sqrt{2}\sin t; \ x = 2\sqrt{2}\cos t.$$

$$\Rightarrow \frac{dT}{dt} = -4\sin^2 t + 4\cos^2 t.$$

$$= -4\sin^2 t + 4\cos^2 t + (4\cos^2 t - 4\cos^2 t).$$

$$\Rightarrow -4(\sin^2 t + \cos^2 t) + 8\cos^2 t.$$

$$= -4 + 8\cos^2 t.$$

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But $\cos 2t = 1 - 2\sin^2 t$
$= 2\cos^2 t - 1.$
$\Rightarrow \cos^2 t = \frac{1+\cos 2t}{2}.$

$$\Rightarrow \frac{dT}{dt} = -4 + 4 + 4\cos 2t = 4\cos 2t.$$

$$\frac{dT}{dt} = 0 \text{ if } \cos 2t = 0. \Rightarrow 2t = \pm\frac{\pi}{2} + k\pi.$$

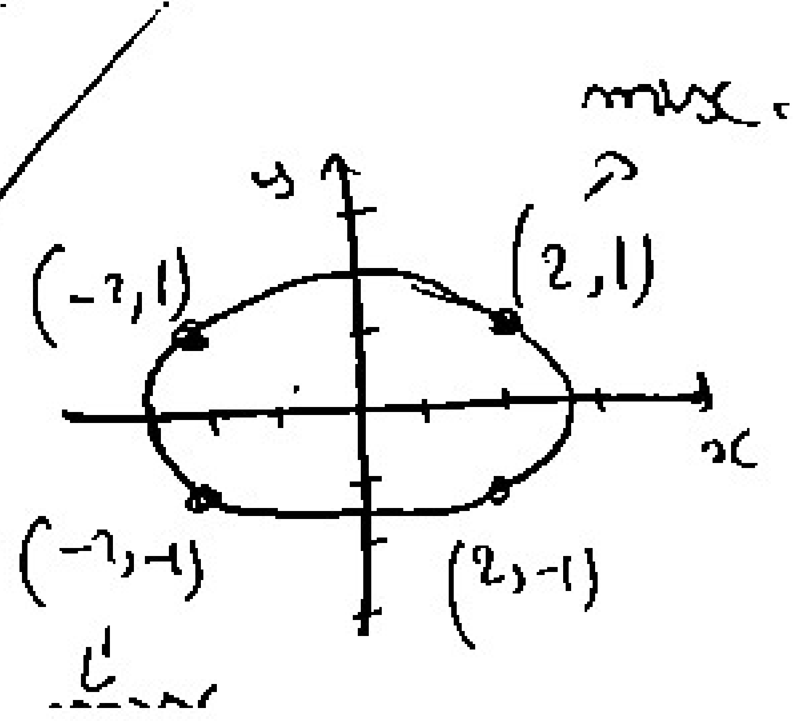
$$\Rightarrow t = \pm\frac{\pi}{4} + \frac{k\pi}{2}.$$

but $0 \le t \le 2\pi$.

$$\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

$$\frac{d^2T}{dt^2} = -8\sin 2t.$$

T	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$-8\sin 2t = \frac{d^2T}{dt^2}$	-8. max	8 min	-8 max	$8 \Rightarrow$ min
Point on the ellipse (x,y)	$(2,1)$	$(-2,1)$	$(-2,-1)$	$(2,-1)$.



7) (15 points) Determine the local maxima, local minima, and saddle points of the function: $f(x,y) = x^3 - 6xy + y^2$.

$f_x = 3x^2 - 6y = 0$

$f_y = -6x + 2y = 0$

$f_x = 3x^2 - 6y = 0$
$f_y = -6x + 2y = 0 \Rightarrow \{ y = 3x \Rightarrow 3x^2 - 18x = 0$

$x(3x - 18) = 0$

$\Rightarrow x = 0$ and $x = 6$.

if $x = 0$ $y = 0$

if $x = 6$ $y = 18$ $\Rightarrow$ critical point $(0,0)$, $(6,18)$.

$f_{xx} = 6x$

$f_{yy} = 2$ $\quad \Delta = 12x - 36$

$f_{xy} = -6$.

$\Rightarrow \Delta|_{0,0} = -36 < 0$

$\Rightarrow$ a saddle point at $(0,0,0)$.

$\Delta|_{6,18} = 72 - 36 =$

$= 36 > 0$, and $f_{yy} > 0 \Rightarrow$ this is a local min at $(6,18,-\cancel{...})$

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