

a) (5 points) Find an equation for the level surface of this function that passes through the point (2,1,1).

$A(2,1,1)$

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

$$= \frac{x}{2} \vec{i} - 2y \vec{j} + 2z \vec{k}$$

$$\vec{\nabla} f_A = \vec{i} - 2\vec{j} + 2\vec{k}$$

equation of level surface $f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$

$$1(x-2) + (-2)(y-1) + (2)(z-1) = 0$$

$$x - 2 - 2y + 2 + 2z - 2 = 0$$

$$x - 2y + 2z - 2 = 0$$

$$f(x,y,z) = c$$

$$\frac{x^2}{4} - y^2 + z^2 = c$$

$$\frac{x^2}{4c} - \frac{y^2}{c} + \frac{z^2}{c} = 1$$

b) (5 points) Identify then draw this level surface.

$$x - 2y + 2z - 2 = 0$$

$$2z = 2y - x + 2$$

$$z = y - \frac{x}{2} + 1$$

$D_x = \mathbb{R}$

Solution on other page.

$x^2 + y^2 = 4$ where center is the origin and radius = 2.
 $\frac{x^2}{4} + \frac{y^2}{4} = 1$

b) (2 points) Determine the boundary of the domain

The boundary of the domain is the disc or the circle centered at $(0,0)$ and has $r=2$.

c) (4 points) Is the domain closed, open, or neither? Justify your answer

it is an open domain since the domain does not include the boundary, which is the disc.

d) (2 points) Is the domain bounded or unbounded? Justify your answer

The domain is unbounded since it cannot be contained in a closed disc.

d) (3 points) Find the function's range

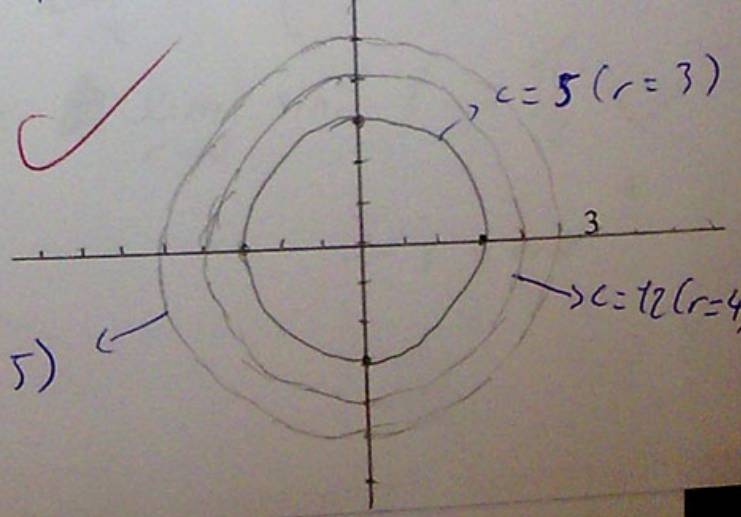
for $x^2 + y^2 = 4^+$ $\Rightarrow \ln(x^2 + y^2 - 4) = -\infty$
 for $x^2 + y^2 = \infty$ $\Rightarrow \ln(x^2 + y^2 - 4) = +\infty$
 So the range is $(-\infty, +\infty)$.

e) (4 points) Describe the function's level curves, and sketch a few of them.

$f(x,y) = \ln(x^2 + y^2 - 4) = c$

$x^2 + y^2 - 4 = e^c$
 $x^2 + y^2 = e^c + 4$
 $\frac{x^2}{e^c + 4} + \frac{y^2}{e^c + 4} = 1$

The level curves are circles centered at the origin with radius $r = \sqrt{e^c + 4}$



We use the 2-path test.

$$y = mx^4$$

$$\Rightarrow \frac{mx^4 \cdot x^4}{x^8 + mx^8} = \frac{mx^8}{x^8(1+m)} = \frac{m}{1+m} \quad \checkmark$$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y}{x^8 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{m}{1+m}$ The limit does not exist since it depends on m and will differ for different values of m .

b) (8 points) Consider the function $f(x,y) = \sin^{-1}\left(1 + \frac{x^4 + y^4}{x^2 + y^2}\right)$

If possible, define $f(0,0)$ in a way that extends f to be continuous at the origin.

~~$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \sin^{-1}\left(1 + \frac{x^4 + y^4}{x^2 + y^2}\right)$~~

~~$= \lim_{(x,y) \rightarrow (0,0)} \sin^{-1}\left(\frac{x^2 + y^2 + x^4 + y^4}{x^2 + y^2}\right)$~~

~~$= \lim_{(x,y) \rightarrow (0,0)} \sin^{-1}(1 + x^2)$~~

we use polar coord.

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ x^2 + y^2 &= r^2 \end{aligned}$$

~~$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$~~

$$\lim_{(x,y) \rightarrow (0,0)} \sin^{-1}\left(1 + \frac{r^4 \cos^4 \theta + r^4 \sin^4 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}\right)$$

$$= \lim_{(x,y) \rightarrow (0,0)} \sin^{-1}\left(1 + \frac{r^4}{r^2}\right)$$

$$= \lim_{r \rightarrow 0} \sin^{-1}(1 + r^2)$$

$$= \lim_{r \rightarrow 0} \sin^{-1} 1 = \frac{\pi}{2} \quad \checkmark$$

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$\pi/3$ cont.

5) (14 points) Let $w = f(x, y)$ be a function of x and y where $x = r^2 + s^2$ and $y = \frac{s}{r}$. Find each of the following and express your answer in terms of $r, s, f_x, f_y, f_{xx}, f_{xy}$, and f_{yy} as needed. Assume that the necessary derivatives exist and that the functions and derivatives are all continuous.

$$w_s = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial w}{\partial y} \cdot \frac{dy}{ds}$$

a) (6 points) Find w_s

$\frac{dx}{dr} = 2r$ ~~$\frac{dx}{ds} = 2s$~~ $= f_x \cdot \frac{dx}{dr} + f_y \cdot \frac{dy}{dr}$
 $\frac{dy}{dr} = \frac{-s}{r^2}$
 ~~$w_s = f_x \cdot 2r + f_y \cdot s$~~
 $w_s = f_x \cdot 2s - f_y \cdot \frac{1}{r^2}$

b) (8 points) Find w_{ss}

$$w_{ss} = \frac{\partial^2 w}{\partial s^2} = \frac{\partial^2 w}{\partial x^2} \cdot \left(\frac{dx}{ds}\right)^2 + \frac{\partial^2 w}{\partial y^2} \cdot \left(\frac{dy}{ds}\right)^2$$

$$= f_{xx} \cdot x_{rr} + f_{yy} \cdot y_{rr}$$

$x_r = 2r$ ~~$x_{rr} = 2$~~ $y_r = \frac{-s}{r^2}$
 $x_{rr} = 2$ $y_{rr} = \frac{2s}{r^3}$

$w_{ss} = 2f_{xx} + \frac{2s}{r^3} \cdot f_{yy}$

- 6) (9 points) Estimate how much the value of $f(x, y, z) = e^x \sin y - z$ will change if the point $P(x, y, z)$ moves 0.2 units from $A(0, 0, 2)$ straight toward $B(1, 2, 0)$.

$$\frac{df}{ds} = \vec{\nabla} f_P \cdot \vec{u}$$

where

$$ds = 0.2$$

$$df = \vec{\nabla} f_P \cdot \vec{u} \cdot ds$$

$$A(0, 0, 2)$$

$$B(1, 2, 0)$$

$$\vec{AB} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$\vec{u} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{1+4+4}} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$$

$$f(x, y, z) = e^x \sin y - z$$

$$f_x = e^x \sin y$$

$$f_y = e^x \cos y$$

$$f_z = -1$$

~~replace~~ replace in A

$$\Rightarrow f_x = 0$$

$$f_y = 1$$

$$f_z = -1$$

$$\Rightarrow \vec{\nabla} f_A = \vec{j} - \vec{k}$$

$$\vec{\nabla} f_A \cdot \vec{u} = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right) + (-1)\left(-\frac{2}{3}\right)$$

$$= 0 + \frac{2}{3} + \frac{2}{3}$$

$$= \frac{4}{3}$$

$$df = \frac{4}{3} (0.2)$$

$$= \frac{4}{15} \text{ units} \approx 0.26 \text{ units}$$

$z + e^{yz} + \ln(x^2 y) = 1 + \ln 8$ defines z implicitly as a differentiable function of x and y , then $\frac{\partial z}{\partial x}$ at $(2, 2, 0)$ will have the value:

- a) $\frac{2}{3}$ b) $\frac{1}{2}$ c) $-\frac{2}{3}$

d) $-\frac{1}{2} \frac{dz}{dx} (yz^2)$

Question B

$f(x, y) = x \cos y + e^{xy}$ decreases most rapidly at $P(1, 0)$ in the direction of:

- a) $\vec{i} + \vec{j}$
 b) $\vec{i} - \vec{j}$
 c) $-\vec{i} - \vec{j}$
 d) $-\vec{i} + \vec{j}$

$\vec{\nabla} f = (\cos y + ye^{xy})\vec{i} + (-x \sin y + xe^{xy})\vec{j}$
 $= (1 + 0)\vec{i} + (0 + 1)\vec{j}$
 $= \vec{i} + \vec{j}$

Question C

An equation for the plane tangent to the surface $z = 2\sqrt{y-x}$ at $P(1, 2, 2)$ is:

- a) $-x + y - z = -1$
 b) $-x + y = 1$
 c) $x - y = 1$
 d) $x - y - z = -1$

$f(x, y) = 2\sqrt{y-x}$
 $f_x = \frac{-1}{\sqrt{y-x}}$ $f_y = \frac{1}{\sqrt{y-x}}$
 $f_x = -1$ $f_y = 1$

$(y-x)^{1/2}$
 $-\frac{1}{2}(y-x)^{-1/2}$

$-1(x-1) + 1(y-2) - (z-2) = 0$
 $-x + 1 + y - 2 - z + 2 = 0$
 $-x + y - z = -1$

Question D

Consider the function $f(x, y)$ such that $f_x = 8 - 2x^2$ and $f_y = 6y - 7$. Then $f(x, y)$ has:

- a) a local maximum at $(-2, \frac{7}{6})$ and a saddle point at $(2, \frac{7}{6})$;
 b) a local minimum at $(-2, \frac{7}{6})$ and a saddle point at $(2, \frac{7}{6})$;
 c) a saddle point at $(-2, \frac{7}{6})$ and a local minimum at $(2, \frac{7}{6})$;
 d) a saddle point at $(-2, \frac{7}{6})$ and a local maximum at $(2, \frac{7}{6})$;

$8 - 2x^2 = 0$ $6y - 7 = 0$
 $2x^2 = 8$ $6y = 7$
 $x^2 = 4$ $y = \frac{7}{6}$
 $x = \pm 2$

$(2, \frac{7}{6})$ $(-2, \frac{7}{6})$
 $f_{xx} = -4x$ saddle
 $f_{yy} = 6$ min.
 $f_{xy} = 0$

$f_{xx} f_{yy} - f_{xy}^2$
 $= -24x$

12 points) Let x, y, z be **nonzero** numbers ($x \neq 0, y \neq 0, z \neq 0$)

Find the largest and smallest products the numbers x, y , and z can have if $x + y + z^2 = 20$

$$f(x, y, z) = xyz$$

$$g(x, y, z) = x + y + z^2 - 20$$

We use Lagrange multipliers

$$\text{where } \vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\vec{\nabla} f = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$\vec{\nabla} g = \vec{i} + \vec{j} + 2z\vec{k}$$

$$yz\vec{i} + xz\vec{j} + xy\vec{k} = \lambda\vec{i} + \lambda\vec{j} + 2z\lambda\vec{k}$$

$$\begin{cases} yz = \lambda & (1) \\ xz = \lambda & (2) \\ xy = 2z\lambda & (3) \\ g(x) = 0 & (4) \end{cases}$$

We deduce from equations

(1) and (2) that

$$x = y$$

so we replace in (3)

$$\begin{cases} x^2 = 2z\lambda & (5) \\ xz = \lambda & (6) \end{cases}$$

$$\text{in (6)} \Rightarrow x = \frac{\lambda}{z}$$

$$\Rightarrow \text{in (5)} \frac{\lambda^2}{z^2} = 2z\lambda$$

$$\Rightarrow \boxed{\lambda = 2z^3}$$

$$x = \frac{\lambda}{z} = \frac{2z^3}{z} = 2z^2$$

$$z = 8 \Rightarrow \boxed{x = 8}$$

$$\boxed{y = 8}$$

$$-y + z^2 - 20 = 0$$

$$+\frac{\lambda}{z} + z^2 - 20 = 0$$

$$+ \frac{2z^3}{z} + z^2 - 20 = 0$$

$$z^2 + 2z^2 + z^2 - 20 = 0$$

$$z^2 = 20$$

$$z = 4$$

$$= \pm 2$$

so the 2 pts are

$$A(8, 8, 2) \text{ \& } B(8, 8, -2)$$