16.5

Please note that you have 6 questions and 7 pages

Round all your answers to 4 digits after the decimal point.

1) **(17 points)** careful inspection of 70 precast concrete supports to be used in a construction project revealed 28 with hairline cracks.

   a) Estimate the true proportion of supports of this type with cracks in a 98% confidence interval.

$\hat{p} = \frac{28}{70} = 0.4$. $\hat{q} = 0.6$.

$n = 70$ large ~~sample~~ population $\hat{p}$ is normally distributed.

$1 - \alpha = \frac{98}{100} \rightarrow \alpha = 1 - 0.98 = 0.02$.

$\frac{\alpha}{2} = 0.01$.

10

a $(1-\alpha)100$ confidence Interval is given by

$[\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}, \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}]$

$z_{\alpha/2} = z_{0.01} = 2.33$.

So the Interval is $[0.4 - 2.33\sqrt{\frac{0.4 \times 0.6}{70}}, 0.4 + 2.33\sqrt{\frac{0.4 \times 0.6}{70}}$

$[0.2636, 0.5364]$

   b) What is the maximum error of estimate associated with the confidence interval constructed in part (a)?

the max error is

$\beta = z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} = 2.33 \times \sqrt{\frac{0.4 \times 0.6}{70}}$

2.5 $= 0.1366$

c) Suppose it is described to estimate the true proportion of cracked supports to within 0.1 units with confidence coefficient 0.98. How many supports should be sampled to achieve the described accuracy?

$B \leq 0.1$ $\quad$ $Z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} \leq 0.1$

$\frac{Z^2_{\alpha/2} \times \hat{p}\hat{q}}{0.1^2} \leq n$, So

4)

10 2) **(10 points)** Items under inspection are subject to two types of defects. About 75% of the items in a large lot are judged to be defect-free. Whereas 15% have a type *A* defect alone and 10% have a type *B* alone. (None have both types of defects). If 5 of these items are randomly selected from the lot, find the probability that 3 have no defects, 1 has a type *A* defect, and 1 has a type *B* defect.

It's a multinomial case with $k=3$.

let X_1 be a R. V which represent ErrorType A.

" X_2 " " " " " " " B.

& X_3 " " " " " no defect.

$n=5$; $P(X_1=1, X_2=1, X_3=3) =$

$\frac{5!}{1! \times 1! \times 3!} \times 0.15^1 \times 0.1^1 \times 0.75^3 = 0.1266$

10

3) (14 points) the daily water demands for a city pumping station exceed 500000 gallons with probability only 0.15. Over a 100-day period, find the probability that demand for over 500000 gallons per day occurs no more than 10 times.

X is a R.V which represent The exceedance 500000

It's a Benomial case with $n = 100$

$P = P(\text{exceed } 500000) = 0.15$

$P(X \le 10) = ?$

$U_X = nP = 100 \times 0.15 = 15$

$V(x) = nPq = 15 \times 0.85 = 12.75$

$\delta_x = 3.5707$

Since 100 is big The population is large and X approximately normal distribution with $U_x = 15$ and $\delta_x = 3.57$

$P(X \le 10) = P\left(Z \le \frac{10 - 15}{3.5707}\right) = P(Z \le -1.42)$

$= 0.5 - P(0 < Z < 1.42) = 0.5 - 0.4222$

$= 0.0778$

18.5

4) **(19 points)** Customers often complain about long waiting times at restaurants before the food is served. A restaurant claims that, the average time taken to serve food to its customers after the order is placed is 15 minutes or less. A local newspaper journalist wanted to check the restaurant's claim at $\alpha = 0.025$. A sample of 36 customers showed that the mean time taken to serve food to them was 15.8 minutes with a standard deviation of 2.5 minutes. Based on the results of this sample the journalist said that the restaurant's claim is not true.

a) Using $\alpha = 0.025$, do you thing that the journalist's conclusion was fair to the restaurant?

What's μ?

$n = 36$ $\quad \bar{X} = 15$ $\quad S = 2.5$

n is large so $\bar{X}$ is normaly distributed.

$H_0 : \mu \leq 15$

$H_1 : \mu > 15$

It's a ~~one~~ one sided test with $\alpha = 0.025$.

~~So $X_c = Z_{\alpha/2}$~~

12.5

~~$= Z_{0.0125}$~~

$X_R = Z_\alpha = Z_{0.025} = 1.96$

acceptance region | Rejected region

X_R

O.V.T.S $= Z_{\bar{X}} = \dfrac{15.8 - 15}{\dfrac{2.5}{\sqrt{36}}} = ~~11.52~~ \; 1.92$

in the acceptance region, so we accept H_0.

and $\mu \leq 15$ is ~~Right~~ accepted

So the journalist's conclusion wasn't fair.

b) Find the p-value of the test. Based on the p-value, would you reject the claim of the restaurant at $\alpha = 0.05$? Explain.

$P\text{value} = P(Z > 1.92) = 0.5 - P(0 < Z < 1.92)$

$= 0.5 - 0.4726 = 0.0274$

6/14

Pvalue is the value under which we accept H_0

$\alpha = 0.05 >$ Pvalue so the claim would be rejected

14

5) **(14 points)** a production process is supposed to be producing special type of resistors. It is claimed that the resistances of the produced resistors have a standard deviation of no more than 0.4 ohm. 15 randomly selected resistors showed a standard deviation of 0.5 ohm. Assuming that the resistances are normally distributed, can the claim be refuted at the 10% significance level?

$n = 15 \quad s = 0.5.$

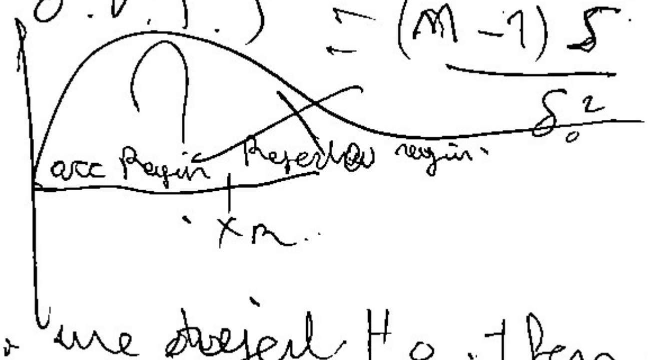
$\frac{(n-1)s^2}{\sigma^2}$ has a χ^2 dist. with $V = 15-1 = 14$.

$H_0: \sigma^2 \leq 0.16.$ It's a one sided test:

$H_a: \sigma^2 > 0.16.$ $\alpha = 0.1.$

So. $X_R = \chi^2_\alpha = \chi^2_{0.1} = 21.0642.$

$O.V.T.S = \frac{(n-1)s^2}{\sigma_0^2} = \frac{14 \times 0.5^2}{0.16} = 21.875$



in the rejection Region

So we reject H_0. then: yes the claim can be refuted at the 10% significance level.

26

6) ~~(26 points)~~ Let X_1, X_2 have the joint probability density function given by

$$f(x_1,x_2)=\begin{cases} x_1+x_2 & \text{for } 0\le x_1\le 1, \quad 0\le x_2\le 1 \\ 0 & \text{elsewhere.}\end{cases}$$

a) Find the marginal density function of X_1 and X_2.

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1,x_2)\,dx_2 = \int_0^1 (x_1+x_2)\,dx_2 = \left[x_1x_2+\frac{x_2^2}{2}\right]_0^1 = x_1+\frac{1}{2}$$

So The marginal prob density function of X_1 ~~and X_2~~

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$$f_1(x_1) = \begin{cases} x_1+\frac{1}{2} & 0<x_1<1 \\ 0 & \text{elsewhere}\end{cases}$$

Same for X_2: $f_2(x_2) = \begin{cases} x_2+\frac{1}{2} & 0<x_2<1 \\ 0 & \text{elsewhere}\end{cases}$

b) Are X_1 and X_2 independent? Explain.

$$f(x_1)\times f(x_2) = \begin{cases} (x_1+\frac{1}{2})(x_2+\frac{1}{2}) & 0<x_1<1, 0<x_2<1 \end{cases}$$

4

$$= \begin{cases} x_1x_2+\frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{4} & 0<x_1<1,\ 0<x_2<1 \\ 0 & \text{elsewhere}\end{cases}$$

$\neq f(x_1,x_2)$ So X_1 and X_2 are dependent

4

c) Find $P\left(X_1\le\frac{1}{2}\middle|X_2\le\frac{1}{2}\right)$.

$$P\left(x_1\le\frac{1}{2} / x_2\le\frac{1}{2}\right) = \frac{P\left(x_1\le\frac{1}{2},\ x_2\le\frac{1}{2}\right)}{P\left(x_2\le\frac{1}{2}\right)}$$

$$P\left(x_1\le\frac{1}{2},\ x_2\le\frac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2}(x_1+x_2)\,dx_1\,dx_2 = \int_0^{1/2}\left(\frac{x_1^2}{2}+x_2x_1\right)\Big|_0^{1/2} dx_2$$

$$= \int_0^{1/2} \left(\frac{1}{8} + \frac{x_2}{2}\right) dx_2 = \left[\frac{1}{8} x_2 + \frac{x_2^2}{4}\right]_0^{1/2}$$

$$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$$P\left(X_2 \leq \frac{1}{2}\right) = \int_0^{1/2} \left(x_2 + \frac{1}{2}\right) dx_2 = \left[\frac{x_2^2}{2} + \frac{1}{2} x_2\right]_0^{1/2}$$

$$= \left[\frac{1}{8} + \frac{1}{4}\right] = \frac{3}{8}$$

$$\text{So } P\left(X_1 \leq \frac{1}{2} \,\middle|\, X_2 \leq \frac{1}{2}\right) = \frac{1/8}{3/8} = \frac{1}{3}$$

8// d) Find $P\left(X_1 \leq \frac{1}{2} \middle| X_2 = 0\right)$

$$f(x_1 \,|\, X_2 = 0) = \begin{cases} \dfrac{f(x_1, 0)}{f_2(0)} & 0 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2x_1 & 0 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$P\left(X_1 < \frac{1}{2} \,\middle|\, X_2 = 0\right) = \int_0^{1/2} f(x_1 \,|\, x_2 = 0)\, dx_1$$

$$= \int_0^{1/2} 2x_1 \, dx_1 = \left[x_1^2\right]_0^{1/2} = \frac{1}{4}$$