GRADE 82

Please note that you have 6 questions and 8 pages

Round all your answers to 3 digits after the decimal point.

1) **(10 points)** From a group of 6 people of which 4 are male, 3 are to be selected randomly. If the random variable *X* represents the number of females among the selected three people.

9

a) Find the probability distribution of *X*.

4 male, 2 female. 3 are chosen randomly.

There are: $C^6_3 = 20$ ways to choose 3 out of 6

X: RV to represent # of females X. 0, 1, 2.

$$p(X=0) = \frac{C^2_0 C^4_3}{C^6_3} = \frac{4}{20} = \frac{1}{5}$$

$$p(X=1) = \frac{C^2_1 C^4_2}{20} = \frac{6\times 2}{20} = \frac{12}{20} = \frac{3}{5}$$

$$p(X=2) = \frac{C^2_2 C^4_1}{20} = \frac{4}{20} = \frac{1}{5}$$

$$p(X=0) + p(X=1) + p(X=2)$$
$$= \frac{1}{5} + \frac{3}{5} + \frac{1}{5}$$
$$= 1 \Rightarrow \boxed{\Sigma p(X=x) = 1}$$

prob. dist. of X ??

-1

b) What is the probability that at least 1 female will be selected? .

$$p(X \geq 1) = 1 - p(X<1)$$
$$= 1 - p(X=0)$$
$$= 1 - \frac{1}{5}$$
$$\boxed{p(X \geq 1) = \frac{4}{5}}$$

at least one female will be selected

2) **(12 points)** The number of customers come to a mini-market flow a Poisson distribution with a mean of 10 customers/hour.

10

a) What is the probability that in an hour, more than 15 customers will come to the mini-market?

$\lambda = 10$ customers/ hour

X. RV to represent # of customers

$P(X \geq 15) = 1 - P(X < 15)$

$= 1 - P(X \leq 14)$

$= 1 - 0.917$ (using table 3)

$\boxed{P(X \geq 15) = 0.083}$

3

b) If the mini-market opens at 8:00 AM, what is the probability that the first customer will not come before 8:05 AM?

Y. exponential R.V to represent the time needed for the first customer to come.

$\beta = \frac{1}{\lambda} = \frac{1}{10}$

5 min $\rightarrow \frac{1}{12}$ hr

7

$P(Y \geq \frac{1}{12}) = e^{-\frac{1}{12}/\frac{1}{10}} = e^{-10/12} = \boxed{0.435}$

18

3) (18 points) In a factory for producing capacitors, the resistances of the produced capacitors are independent normally distributed with a mean of 30 ohm, and a standard deviation of 1 ohm.

a) If a capacitor is randomly selected, what is the probability that its resistance will be greater than 29 ohm?

$\mu = 30$ ohm $\quad S = 1\,\Omega$

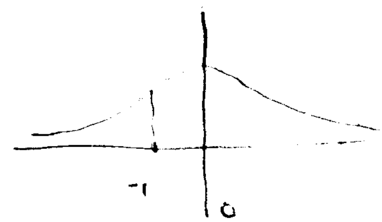
X: normal R.V

$P(X > 29) = P\left(Z > \frac{29-30}{1}\right) = P(Z > -1)$

$\rightarrow P(Z > -1) = P(-1 < Z < 0) + 0.5$

$= 0.3413 + 0.5$

$\boxed{P(Z > -1) = 0.8413} = P(X > 29)$



b) If 5 capacitors are selected randomly, what is the probability that at least 2 of them, will have each a resistance greater than 29 ohm?

We have a binomial case: $n = 5$

Y: binomial R.V $\quad p = 0.8413 \quad q = 1 - p = 0.1587$

$P(Y \geq 2) = 1 - P(Y < 2)$

$= 1 - P(Y=0) - P(Y=1)$

$= 1 - [C^5_0 p^0 q^5] - [C^5_1 p q^4]$

$= 0.999 \quad [C^5_1 p q^4]$

$\boxed{P(Y \geq 2) = 0.996}$

c) If we are testing the capacitors one by one, what is the probability that the 5th tested one, will be the 2nd one with resistance greater than 29 ohm?

We have a negative binomial case:

Z: negative binomial R.V with $n=5$ $r=2$ $p=0.8413$ $q=0.1587$

$P(Z=5) = C_1^4 p^2 q^3$

$= 4\,(0.8413)^2\,(0.1587)^3$

$= 4 \times 0.708 \times$

$\boxed{P(Z=5) = 0.011}$

4

d) If 2 such capacitors are selected, what is the probability that the sum of their resistances is less than 58 ohm?

X_1: resistance of the first capacitor $\mu_1 = 30$ $\sigma_1 = 1$

X_2: " " 2nd " $\mu_2 = 30$ $\sigma_2 = 1$

6

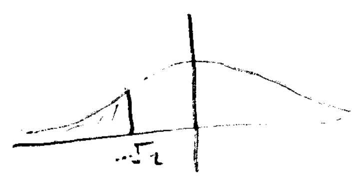
Y: sum of the resistances of the two: $\mu_Y = 60$ $\sigma_Y = \sqrt{1^2+1^2} = \sqrt{2}$

$\boxed{Y = X_1 + X_2}$ Y is normally distributed.

$P(Y<58) = P\left(Z<\frac{58-60}{\sqrt{2}}\right) = P\left(Z<\frac{-2}{\sqrt{2}}\right) = P(Z<-\sqrt{2})$

$\rightarrow P(Z<-\sqrt{2}) = 0.5 - P(-\sqrt{2}<Z<0)$

$= 0.5 - 0.4207$



$\boxed{P(Z<-\sqrt{2}) = 0.079}$

15

4) **(16 points)** In a firm for repairing TV sets, the total cost in \$ for repairing a TV is a random variable Y, defined as the sum of the labor cost which is an exponential random variable with mean of \$ 10, and the spare-parts cost which is a gamma random variable with $\alpha = 2$ and $\beta = 10$.

a) Find the mean, variance, and the probability density function of Y.

Let X be the exp R.V $\quad \beta = 10. \quad \alpha_x = 1$

Z " the gamma R.V $\quad \alpha_z = 2 \quad \beta = 10$

$Y = X + Z$ $\qquad$ Y has gamma distribution

$Y. \quad \alpha_y = \alpha_x + \alpha_z = 3 \qquad \beta = 10$

$$\boxed{E(Y) = \alpha_y \beta = 3 \times 10 = 30}$$

$$\boxed{V(Y) = \alpha \beta^2 = 3 \times 100 = 300}$$

10/3

$\boxed{\text{p.d.f}}$

$$f(y) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

replacing by the above terms we get:

$$f(y) = \frac{1}{(10)^3 (2!)} y^2 e^{-y/10} = \frac{1}{2000} y^2 e^{-y/10}$$

b) What is the probability that the total cost of repairing a TV will exceed \$ 40?

~~P(Y > 40):~~ We use ~~Tchebycheff's theorem~~ ~~$SD = \sqrt{V(Y)} = 17.32$~~

~~Letting $\mu + K\sigma = 40$~~

~~$30 + K \times 17.32 = 40$~~

~~$17.32 K = 10$~~

~~$K = 0.577$~~

~~$P(Y > 40) =$ at most $\frac{1}{K^2}$~~

~~$= \frac{1}{}$~~

$$P(Y>40) = \int_{40}^{\infty} \frac{1}{2\times 1000} x^2 e^{-x/10} dx$$

x^2	$e^{-x/10}$
$2x$	$-10e^{-x/10}$
2	$100e^{-x/10}$
0	$-1000e^{-x/10}$

$$= \frac{1}{2}\left[\frac{-10x^2e^{-x/10}}{1000} - \frac{200xe^{-x/10}}{1000} - \frac{2000e^{-x/10}}{1000}\right]_{40}^{\infty}$$

$$= \frac{1}{2}\left[\frac{-x^2e^{-x/10}}{100} - \frac{xe^{-x/10}}{10} - 2e^{-x/10}\right]_{40}^{\infty}$$

$$= \frac{1}{2}\left[\frac{1600e^{-4}}{100} + \frac{40e^{-4}}{10} + 2e^{-4}\right]$$

$$= \frac{1}{2}\left[16e^{-4} + 8e^{-4} + 2e^{-4}\right]$$

$$= 11e^{-4} \quad = 13e^{-4}$$

$$\boxed{P(Y>40) = 0.201}$$

$$= 0.238$$

20

5) (20 points) Suppose that two random variables X and Y have the joint probability density function $f(x,y)=\begin{cases} kx(1-y) & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

a) Find k.

$$k\int_0^1\int_0^1 x(1-y)\,dx\,dy = k\int_0^1\int_0^1 (x-xy)\,dx\,dy$$

$$= k\int_0^1 \frac{x^2}{2} - \frac{x^2}{2}y \Big|_0^1 dy$$

$$= k\int_0^1 \left(\frac{1}{2} - \frac{1}{2}y\right) dy$$

$$= k\left[\frac{1}{2}y - \frac{y^2}{4}\right]_0^1$$

$$= k\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$= k\left[\frac{1}{4}\right] = 1$$

$$\longrightarrow \boxed{k=4}$$

3/

b) Find $P\left(X \le \frac{1}{2},\ Y \le \frac{1}{2}\right)$.

$$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = 4\int_0^{1/2}\int_0^{1/2} (x-xy)\,dx\,dy$$

$$= 4\int_0^{1/2} \frac{x^2}{2} - \frac{x^2}{2}y \Big|_0^{1/2} dy$$

$$= 4\int_0^{1/2} \left(\frac{1}{8} - \frac{1}{8}y\right) dy = 4\left[\frac{1}{8}y - \frac{1}{16}y^2\right]_0^{1/2}$$

$$= 4\left[\frac{1}{16} - \frac{1}{64}\right]$$

$$= 3/16$$

5/

c) Find the marginal density function of X.

$f(x) = \int_0^1 f(x,y)dy = 4\int_0^1 (x - xy)dy = 4\left[xy - \frac{xy^2}{2}\right]_0^1$

$f(x) = 4\left(x - \frac{x}{2}\right) = 4\frac{x}{2} = 2x$

3/3

d) Are X and Y independent? Explain.

we have $f(x,y) = 4x(1-y)$ and $f(x) = 2x$ so actually $f(x,y)$ can be written as $f(x) \cdot f(y)$ where $f(y) = 2(1-y)$ and hence X & Y are independent.

3/3

e) Find $P\left(X \le \frac{1}{2} \middle| Y \ge \frac{1}{2}\right)$.

$P(X \le \frac{1}{2} / Y \ge \frac{1}{2})$ since X & Y are independent then this reduces to $P(X \le \frac{1}{2}) = \int_0^{1/2} f(x)dx = \int_0^{1/2} 2x\,dx$

$= x^2\Big|_0^{1/2} = \frac{1}{4}$

3/3

f) Find cov.(X,Y).

Since X & Y are independent then their covariance is zero

$cov(X,Y) = 0$.

3/3

10

6) **(10 points)** In a gambling game, using an ordinary deck of 52 playing cards, a man is paid \$ 3 if he draws a jack, a queen, or a king, and is paid \$ 9 if he draws an ace. If he draws any other card he should pay \$ x. Find x if we want the game to be fair. (A fair game means the expected gain is zero)

52 cards

paid 3 \$: jack, queen or King :

9 \$: ace.

loose x \$: any other

~~X_1: number of jack, queen or King. (RV) receive 3 \$~~

~~X_2: number of aces (RV) receive 9 \$~~

~~X_3: number of others. (RV) : pay x \$~~

Random variables

G: gain $p(X_1) = \frac{12}{52} = \frac{4j+4q+4K}{52}$ → total cards

X_1 : # of q, j, K

these are not R-variables

$p(X_2) = \frac{4}{52}$ → 4 aces

X_2 : # of ace

$p(X_3) = \frac{36}{52}$

X_3 : # of others

~~$G = X_1 + X_2 - X_3$~~

~~$E(G) =$~~

$\boxed{G = 3X_1 + 9X_2 - xX_3}$

$E(G) = 3E(X_1) + 9E(X_2) - xE(X_3) = 0$

for the game to be fair.

$E(X_1) = np_1 = \boxed{12} = 52 \times \frac{12}{52}$

$E(X_2) = np_2 = \boxed{4} = \frac{4}{52} \times 52$

$E(X_3) = \boxed{36} = \frac{36}{52} \times 52$

thus ~~$xE(X_3) = 36 \times 36$~~

~~$x \times 36 = 72$~~

$\boxed{x = 2 \$}$

$3 \times 12 + 9 \times 4 - x \times 36 = 0.$

$36x = 72$

$\boxed{x = 2 \$}$

MAT 326 Exam I

1) (18 points) The proportion of time that an industrial robot is in operation during a 40 – hour week is a random variable X with density function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

8

a) What value of k makes $f(x)$ a probability density function?

b) For the robot under study, the profit Y/week is given by $Y = 180X - 60$. Find $E(Y)$ and $V(Y)$.

c) Find an interval in which the profit should lie for at least 75% of the weeks.

(a) Since Pdf $\Rightarrow \int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 k\cdot x\,dx = 1$

$k\frac{x^2}{2}\Big|_0^1 = 1 \quad \frac{k}{2} = 1 \Rightarrow \boxed{k = 2}$ 2

(b) $Y = 180X - 60$

$E(Y) = 180\,E(X) - 60$ 2

where $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

where (each unit of time = 40 hrs) $\Rightarrow E(X) = \frac{2}{3}(40) = \frac{80}{3}$ hrs.

$\Rightarrow E(Y) = 180\left(\frac{2}{3}\right) - 60 = \boxed{60 = E(Y)}$ 3 8

$V(Y) = 180^2\,V(X)$

where $V(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 2x^3\,dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$

2

$\therefore V(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$

3

$\therefore V(Y) = 180^2\left(\frac{1}{18}\right) = 1800$ $\boxed{V(Y) = 18\ldots}$

(c) According to Tchebycheff's theorem

$P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}$ $\quad 1 - \frac{1}{k^2} = 0.75$

$\frac{1}{k^2} = 0.25$

2

interval $\mu - k\sigma \le x \le \mu + k\sigma$

2) **(8 points)** An expert sharpshooter misses a target 10 percent of the time. Find the probability, that she will miss the target for the second time on the tenth shot.

$p(\text{miss}) = 0.1$

$p(\text{hit}) = 0.9$

Negative-binomial case where $r = 2$ & $x = 10$

$\therefore p = C_{r-1}^{x-1} p^r q^{x-r} = C_1^9 (0.1)^2 (0.9)^8 = 9\cancel{(0.1)^2}(0.9)^2$

$= \boxed{0.0729}$

$= 0.0387$

7

3) **(14 points)** The number of planes arriving per day (24 hours) at certain airport is a random variable having a Poisson distribution with mean equals 20 airplanes.

a) What is the probability that on a random day no more than 12 airplanes will arrive at this airport?
b) What is the probability, that the time between successive arrivals is less than 1 hour?
c) What is the mean time between successive arrivals?

14

Poisson Dist $\lambda = 20$

~~a) $p(x \leq 20) = 0.47$ (using table 3)~~

a) $p(x \leq 12) = 0.039$ (using table 3) 4

b) Time b/w successive arrivals is an exponential distribution Y with: $\beta = \frac{1}{\lambda}$ if $t = 24$ hrs $\lambda = 20$

$\therefore$ in 1 hour period

$\lambda = \frac{20}{24} = \frac{5}{6}$ arrivals per hour

24 hrs $\to \lambda = 20$

$\therefore Y$ has $\beta = \frac{1}{\lambda} = \frac{6}{5}$ hrs b/w successive arrivals

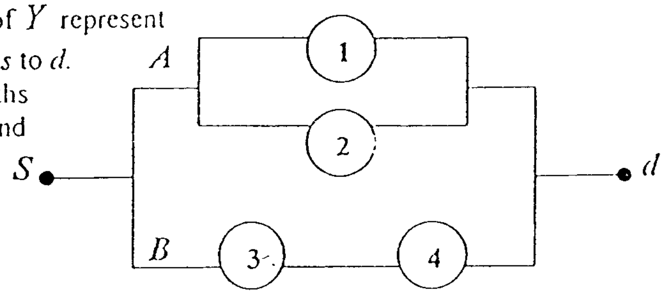
$\to t = 1$ hr $\lambda = \frac{20}{24}$

$p(Y < 1) = 1 - p(Y \geq 1) = 1 - e^{-x/\beta} = 1 - e^{-1/6/5}$

$= 1 - e^{-5/6} = 0.5654$

c) mean $= \beta = \frac{6}{5}$ hrs $= 1.2$ hrs. 3

4) (15 points) Consider the system of water shown flowing through valves from s to d. Valves 1, 2, 3, and 4 operate independently and each correctly opens on signal with probability 0.8. Let the values of Y represent the number of open paths from s to d. (Note that, there is only two paths the first through A and the second through B). Find $p(Y=1)$.



$p(\text{open}) = 0.8$

$p(Y=1) = ?$

i.e each is equally probable of prob $\frac{1}{16}$

$p(Y=1) = \frac{1}{16}(10) = \frac{10}{16}$

$= \frac{5}{8}$

if $1 \Rightarrow$ on
$0 \Rightarrow$ off

V_1	V_2	V_3	V_4	
0	0	0	0	$Y=0$
0	0	0	1	$Y=0$
0	0	1	0	$Y=0$
0	0	1	1	$Y=1$
0	1	0	0	$Y=1$
0	1	0	1	$Y=1$
0	1	1	0	$Y=1$
0	1	1	1	$Y=2$
1	0	0	0	$Y=1$
1	0	0	1	$Y=1$
1	0	1	0	$Y=1$
1	0	1	1	$Y=2$
1	1	0	0	$Y=1$
1	1	0	1	$Y=1$
1	1	1	0	$Y=1$
1	1	1	1	$Y=2$

5) (15 points) In a certain assembly plant, three machines, B_1, B_2, B_3 make 30%, 45%, and 25%, respectively, of the product. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective.

a) Suppose that a finished product is randomly selected. What is the probability that it is defective?

b) If a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

15

	B_1	B_2	B_3
	30%	45%	25%
def	2%	3%	2%

~~(a) p(def) = (2+3+2)% = 7% = 0.07~~

(a) $100\% \xrightarrow{B_1} 30\%$, $100\% \xrightarrow{B_2} 45\%$, $100\% \xrightarrow{B_3} 25\%$

or

2% of out of 30% are def

∴ $2 \times 0.3 = 0.6\%$

3% out of 45% are def

$3 \times 0.45 = 1.35\%$

2% out of 25 are def

$2 \times 0.25 = 0.5\%$

∴ p(1 def out of 100) = $0.5 + 0.6 + 1.35(\%)$ = 2.45%

~~p(def) = 2.45%~~

$\boxed{p(\text{def}) = 0.0245}$

(b) $p(x \in B_3 / x \text{ is def}) = \dfrac{p(\text{intersection})}{p(x \text{ is def})}$

$= \dfrac{0.02 \times 0.25}{0.0245} = \boxed{0.204}$

6) (15 points) If a company employs n salespersons, its monthly gross sales in hundreds of dollars is a random variable X having a gamma distribution with $\alpha = 40\sqrt{n}$ and $\beta = 2$.

How many salespersons, the company should employ in order to maximize the expected monthly profit, if you know that the sales cost is \$800 per salesperson?

X: gamma dist $\alpha = 40\sqrt{n}$ $\beta = 2$

$n = ?$ E(profit) = max

\$800 / person.

6

$\text{Profit} = P = n(X - 800) = X - 8n$

$E(P) = n\,E(X) - 800n = 80n^{3/2} - 800n$

$\frac{dE(P)}{dn} = 80\left(\frac{3}{2}\right)n^{1/2} - 800 = 0$ (max)

$\Rightarrow 120\sqrt{n} = 800$

$\boxed{n \approx 44 \text{ persons}}$

5 points) A certain type of elevator has a maximum weight capacity X_1, which is normally distributed with a mean and standard deviation of 5000 and 300 pounds, respectively. For a certain building equipped with this type of elevator, the elevator loading, X_2, is a normally distributed random variable with a mean and standard deviation of 4000 and 400 pounds, respectively.
For any given time that the elevator is in use, find the probability that it will be overloaded, assuming X_1 and X_2 are independent.

$X_2 \begin{cases} \mu_2 = 4000 \\ S_2 = 400 \end{cases}$

$\mu_1 = 5000$
$S = 300$

(overloaded)

Elevate is overloaded iff $x_2 > 4000$ pounds since the elevator cannot hold more than 5000 pds. x_2 should be b/w 4000 & 5000 pds.

$P(4000 < \cdot < 5000) = P\left(\frac{4000-4000}{400} < z < \frac{5000-4000}{400}\right)$

$P(0 < 2.5) = 0.4938$

$z_1 = \frac{x_1 - 5000}{300}$ $\quad z_2 = \frac{1000}{400}$