1) **(20 points)** Solve $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

$m^2 + 2m - 8 = 0$

$\Delta = b^2 - 4ac$

$= 4 - 4(-8)(1)$

$= 4 + 32$

$= 36$

$m_1 = \dfrac{-b - \sqrt{\Delta}}{2a} = \dfrac{-2-6}{2a}$ $\qquad m_2 = \dfrac{-b + \sqrt{\Delta}}{2a} = \dfrac{-2+6}{2}$

$m_1 = -4$ $\qquad m_2 = 2$

$y_h = c_1 e^{-4x} + c_2 e^{2x}$

Try $y_p = Axe^{-2x} + Bxe^{-x}$ $\qquad$ Try $y_p = Ae^{-2x} + Be^{-x}$

$y'_p = A(-2xe^{-2x} + e^{-2x}) + B(-(x)e^{-x} + e^{-x})$

18

$y''_p = A(4xe^{-2x} + (-2e^{-2x}) + (2e^{-2x})) + B(-(e^{-x} - xe^{-x}) - e^{-x})$

$= A(4xe^{-2x} - 4e^{-2x}) + B(xe^{-x} - 2e^{-x})$

$4A(xe^{-2x} - 4e^{-2x}) + B(xe^{-x} - 2e^{-x}) + 2A(-2xe^{-2x} + e^{-2x}) +$
$2B(-(x)e^{-x} + e^{-x}) - 8Axe^{-2x} + 8Bxe^{-x} = 2e^{-2x} - e^{-x}$

$e^{-2x}(4Ax - 4A - 4Ax + 2A - 8Ax) + e^{-x}(Bx - 2B - 2Bx + 2B + 8Bx)$

$= 2e^{-2x} - e^{-x}$

$-2A = 2$

$7B = 0$
$B = 0$

$Y = y_h + y_p = c_1 e^{-4x} + c_2 e^{2x} - xe^{-2x} + 0.$

2) **(20 points)** Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$

$$y = x^m$$

$$(m-1)(m) - 3(m) + 3 = 0$$

$$m^2 - m - 3m + 3 = 0$$

$$m^2 - 4m + 3 = 0$$

$(a + b + c = 0)$

$m_1 = 1 \qquad m_2 = 3$

$$Y_h = c_1 x + c_2 x^3$$

Try $y_p = (Ax^4 + Bx^3 + Cx^2 + Dx + F)e^x$

$y'_p =$ ___

$y''_p =$ ___

we replace in the equation

and by identification we get the constants

3) **(20 points)** Solve $y'' + y = \sec x$

$$y'' + y = \frac{1}{\cos x}$$

$$m^2 + 1 = 0 \quad m^2 = -1 \quad m = \pm i$$

$$\alpha = 0 \quad \beta = 1. \quad c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$$

$$m_1 = 1 \quad m_2 = -1$$

$$Y_h = c_1 e^{x} + c_2 e^{-x}$$

$$e^{x} u_1 + u_2 e^{-x} = 0$$

$$e^{x} u_1' - u_2 e^{-x} = \cancel{\sec x} \frac{1}{\cos x}$$

$$w = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = e^{x}(-e^{-x}) - (e^{-x})(e^{x}) = -1 - 1 = -2$$

8

$$w_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{\cos x} & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{\frac{1}{\cos x}} = -e^{-x}\cos x$$

$$u_1' = \frac{w_1}{w} = \frac{-e^{-x}\cos x}{-2} = \frac{1}{2} e^{-x}\cos x$$

$$w_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{1}{\cos x} \end{vmatrix} = \frac{e^{x}}{\cos x}$$

$$u_2' = -\frac{1}{2}\frac{e^{x}}{\cos x}$$

$$\cancel{y = c_1 e^{x} + c_2 e^{-x}} + \cancel{\frac{1}{2}\int e^{x}}$$

4) (20 points) Evaluate:

a) $\mathcal{L}\left((e^t - e^{-t})^2\right)$

b) $\mathcal{L}\left(\begin{cases} 0 & 0 \le t \le 2 \\ e^{2t}+1 & t > 2 \end{cases}\right)$

$e^{2t+1} \begin{cases} 0 & 0 \le t \le 2 \\ 1 & t > 2. \end{cases}$

a) $\mathcal{L}\left((e^t - e^{-t})^{2}\right)$

4 $= \mathcal{L}(e^{2t}) - \mathcal{L}(e^{-2t})$

$\boxed{= \frac{1}{s-2} + \frac{1}{s+2}}$ $-\frac{2}{s}$

b) $\mathcal{L}\left(\begin{cases} 0 & 0 \le t \le 2 \\ e^{2t}+1 & t > 2 \end{cases}\right)$

$\mathcal{L}\left(e^{2t+1} \begin{cases} 0 & 0 \le t \le 2 \\ 1 & t > 2 \end{cases}\right)$

4 $\mathcal{L}\, e^{2t+1} u(t-2) = e^{-2s} \mathcal{L}\, u(t+2)$

c) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4s+5}\right]$

$$\mathcal{L}^{-1}\left[\frac{A}{s^2}+\frac{B}{4s}+\frac{C}{5}\right]$$

$A=\frac{1}{20}$ $B=0$ $C=0$ (Refer to scratch please)

$$=\mathcal{L}^{-1}\left[\frac{1}{20s^2}\right]$$

$$=\frac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]=\frac{1}{20}t.$$

$$\frac{s}{(s+2)^2+1}$$

$s+2 \longrightarrow s$

$$\frac{s-2}{s^2+1}$$

e^{-2t}

$s+2$

5) (20 points) Solve the following IVP by the Laplace method.

$$y'' + 4y = u_{2\pi}(t)\sin(t) = \begin{cases} 0 & t < 2\pi \\ \sin(t) & t \geq 2\pi \end{cases}$$

$y''+4y = u(t-2\pi)\sin t$

$y(0) = 0 \quad y'(0) = 0$

$\mathcal{L}y'' + \mathcal{L}4y = \mathcal{L}u_{(t-2\pi)}\sin(t)$

$\cancel{s^2 y - sy(0) + 4\mathcal{L}y} =$

$s^2\mathcal{L}y - s\,\cancel{y(0)} - \cancel{y'(0)} + 4\mathcal{L}y = e^{-2\pi s}\mathcal{L}\sin(t-2\pi)$

$\mathcal{L}y(s^2+4) = e^{-2\pi s}\dfrac{1}{s^2+1}$

$\mathcal{L}y = e^{-2\pi s}\dfrac{1}{(s^2+4)(s^2+1)}$

$y = \mathcal{L}^{-1}\dfrac{e^{-2\pi s}}{(s^2+4)(s^2+1)} =$

10