NDU

Notre Dame University

MAT 235

Differential Equations

Exam II

$\frac{99\frac{1}{2}}{100}$ Excellent

Tuesday January 12, 2010

Duration: 55 minutes

Name: [redacted]

Section: [redacted]

Instructor: [redacted]

Grade: [redacted]

1) **(15 points)** Solve the differential equation

$$y''' + y'' + y' = 0$$

$y''' + y'' + y' = 0$ Let $y = e^{xm}$

$y' = m e^{xm}$

$y'' = m^2 e^{xm}$

$y''' = m^3 e^{xm}$

$e^{xm}(m^3 + m^2 + m) = 0$

$e^{xm} \neq 0$

$m^3 + m^2 + m = 0$

$m(m^2 + m + 1) = 0$

$m_1 = 0$

$\Delta = b^2 - 4ac$

$= (1)^2 - 4(1)(1)$

$= 1 - 4$

$= -3 = 3i^2$

$m_2, m_3 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 \pm \sqrt{3i^2}}{2}$ → $\frac{-1 + \sqrt{3}i}{2}$, $\frac{-1 + (-\sqrt{3}i)}{2}$

$\alpha = -1/2$

$\beta = \sqrt{3}/2$

$\rightarrow y_h(x) = C_1 e^{0m} + e^{-1/2 x}\left[A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right]$

$\rightarrow y_h(x) = C_1 + e^{-1/2 x}\left[A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right]$

15

2) **(35 points)** Solve the initial-value problem

$$y'' + y = \cos x$$
$$y(0) = 0, y'(0) = 1$$

using the undetermined coefficients method.

$y'' + y = \cos x$

Consider the associated homogeneous diff eqn:

* $y'' + y = 0$

$e^{mx}(m^2+1) = 0$

$\neq 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m^2 = i^2$

$\rightarrow m_1, m_2 = \pm i \quad (\alpha = 0;\ \beta = 1)$

$\rightarrow y_h(x) = A\cos x + B\sin x$

* $y_p(x)$ is of the form $C\cos x + D\sin x$ X

$\rightarrow y_p(x) = \boxed{Cx\cos x + Dx\sin x}$

$y'_p(x) = C[1(\cos x) + (-\sin x)(x)] + D[(1)(\sin x) + (\cos x)(x)]$

$= C[\cos x - x\sin x] + D[\sin x + x\cos x]$

$= C\cos x - Cx\sin x + D\sin x + Dx\cos x$

$y''_p(x) = -C\sin x - C[(1)(\sin x) + (\cos x)(x)] + D\cos x + D[(1)(\cos x) + (-\sin x)(x)]$

$\rightarrow y''_p(x) = -C\sin x - C\sin x - Cx\cos x + D\cos x + D\cos x - Dx\sin x$

$= \boxed{-2C\sin x + 2D\cos x - Cx\cos x - Dx\sin x}$

Substitute obtained values in given ode:

$\rightarrow -2C\sin x + 2D\cos x - \cancel{Cx\cos x} - \cancel{Dx\sin x} + \cancel{Cx\cos x} + \cancel{Dx\sin x} = \cos x$

$\rightarrow -2C\sin x + 2D\cos x = \cos x$

$\rightarrow 2D = 1 \rightarrow D = 1/2$

$-2C = 0 \rightarrow C = 0$

so $\boxed{y_p(x) = \frac{1}{2} x \sin x}$

so $y(x) = y_p(x) + y_h(x)$
general solution

* $\rightarrow y(x) = \frac{1}{2} x \sin x + A\cos x + B\sin x$

But the arbitray csts may be obtained from given initial values:

$y(0) = 0$

$\rightarrow \cancel{\frac{1}{2}(0)\sin(0)} + A\cos(0) + \cancel{B\sin(0)} = 0$

$\boxed{A = 0}$

* $y'(x) = \frac{1}{2}\sin x + \frac{1}{2}x\cos x - A\sin x + B\cos x$

$y'(0) = 1$

$\rightarrow \cancel{\frac{1}{2}\sin(0)} + \cancel{\frac{1}{2}(0)\cos(0)} - \cancel{A\sin(0)} + B\cos(0) = 1$

$\boxed{B = 1}$

so $\boxed{y(x) = \frac{1}{2}x\sin x + \sin x}$

35

3) **(30 points)** Solve the differential equation

$$x^2y'' - xy' + y = x^3 \quad , \quad x > 0$$

The above is of the Cauchy-Euler form (the associated homogeneous part)

$\Rightarrow$ set $y = x^m$

$y' = m x^{m-1}$

$y'' = (m-1)(m) x^{m-2}$

$$x^2(m-1)(m)x^{m-2} - x(m x^{m-1}) + x^m = 0$$

$$x^m(m-1)(m) - x^m(m) + x^m = 0$$

$$x^m[(m-1)(m) - m + 1] = 0$$

$$x^m[m^2 - m - m + 1] = 0$$

$$x^m\underbrace{[m^2 - 2m + 1]}_{\text{characteristic eqn}} = 0$$

$*$ $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m_1 = m_2 = 1$

$\rightarrow$ $\boxed{y_h(x) = \underbrace{Ax}_{y_1(x)} + \underbrace{Bx\ln x}_{y_2(x)}}$

To find $y_p(x)$:

$$y_p(x) = -y_1 \int \frac{y_2\, g\, dx}{W(y_1, y_2)} + y_2 \int \frac{y_1\, g\, dx}{W(y_1, y_2)}$$

but $W(y_1, y_2) = \begin{vmatrix} Ax & Bx\ln x \\ A & B\ln x + B \end{vmatrix} = Ax(B\ln x + B) - ABx\ln x$

$= ABx\cancel{\ln x} + ABx - AB\cancel{x\ln x}$

$= ABx$

Taking ode back to standard form

$$\rightarrow y_p(x) = -Ax\int \frac{\cancel{B}x\ln(x)}{A\cancel{B}\cancel{x}}dx + Bx\ln x\int \frac{Ax \cdot x}{AB\cancel{x}}dx$$

$$= -\cancel{A}x\int \frac{x\ln x}{\cancel{A}}dx + Bx\ln x\int \frac{x}{B}dx$$

$$= -\cancel{A}x\left(\frac{1}{\cancel{A}}\right)\int x\ln x\, dx + \cancel{B}x\ln x\left(\frac{1}{\cancel{B}}\right)\int x\, dx$$

$$= -x\int x\ln x\, dx + x\ln x\int x\, dx$$

$$= -x\int x\ln x\,dx + x\ln x\int x\,dx$$

but $\int x\ln x\,dx$ by parts

Let $u = \frac{x^2}{2}$ & $v = \ln x$

$u' = x$ $\quad v' = \frac{1}{x}$

$$\rightarrow \int u'v\,dx = uv - \int uv'\,dx$$

$$= \left(\frac{x^2}{2}\right)(\ln x) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx$$

$29\frac{3}{4}$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2$$

$$\rightarrow y_p(x) = -x\left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right] + x\ln x\left[\frac{x^2}{2}\right]$$

$$= \cancel{-\frac{x^3}{2}\ln x} - \frac{1}{4}x^3 + \cancel{\frac{x^3}{2}\ln x}$$

$$= +\frac{1}{4}x^3$$

$$\rightarrow y(x) = y_h(x) + y_p(x)$$

$$= Ax + Bx\ln x + \frac{1}{4}x^3$$

4) **(20 points)** Given the differential equation

$$(1-x^2)y'' + 2xy' = 0, \quad -1 < x < 1$$

a) Given that $y_1(x) = 1$ is a solution, find another solution $y_2(x)$ using the reduction of order method.

$$y_2 = y_1 \int \left(\frac{e^{-\int p(x)dx}}{y_1^2} \right) dx$$

but in standard form

$$y'' + \frac{2x}{(1-x^2)} y' = 0$$

$$\rightarrow y_2(x) = 1 \int \frac{e^{-\int \left(\frac{2x}{(1-x^2)}\right) dx}}{1^2} dx$$

$$= 1 \int \frac{e^{\ln(1-x^2)}}{1} dx$$

$$= 1 \int (1-x^2)\, dx$$

$$= \int 1 - x^2 \, dx$$

$$= \int dx - \int x^2 dx$$

$$= \boxed{x - \frac{x^3}{3} + C}$$

b) Check that $y_1(x)$ and $y_2(x)$ are linearly independent.

$$W(y_1 y_2) = \begin{vmatrix} 1 & \frac{x^3}{3} - x \\ 0 & x^2 - 1 \end{vmatrix} = 1(x^2-1) - 0 = \boxed{x^2-1}$$

$* \ (x-1)(x+1) = 0$

but $-1 < x < 1$

$\Rightarrow \boxed{x^2 - 1 \neq 0}$

$\Rightarrow$ The solutions are linealy independent

$19\frac{3}{4}$

c) Deduce the general solution of the given differential equation.

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$
$$= C_1\left(\frac{x^3}{3} - x\right) + 1C_2$$