Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 235
Ordinary Differential Equations

Exam # 2

Monday January 9, 2006

Duration: 55 minutes

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Section: ____________

Instructor: Saleeby

Grade: 99 good!

Directions

1. Write neatly and clearly.
2. Do not use pencils unless for graphing.
3. Show all work.
4. Programmable calculators are not allowed.
5. Turn off your mobile phones.

Laplace Transforms

	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	e^{at}	$\frac{1}{s-a}$
3	$\sin bt$	$\frac{b}{s^2+b^2}$
4	$\cos bt$	$\frac{s}{s^2+b^2}$
5	$\sinh bt$	$\frac{b}{s^2-b^2}$
6	$\cosh bt$	$\frac{s}{s^2-b^2}$
7	$t^n (n=1,2,\ldots)$	$\frac{n!}{s^{n+1}}$
8	$e^{at} f(t)$	$F(s-a)$
9	$u_a(t) g(t-a)$	$e^{-as} G(s)$
10	$\mu_a(t) g(t)$	$e^{-as} L\{g(t+a)\}$
11	$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}$
12	$\mu_a(t)$	$\frac{e^{-as}}{s}$

Please note that you have 5 questions and a total of 8 pages

1) **(20 points)** Solve the differential equation $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x\sin x$, $(x>0)$

$$x^2y'' - 2xy' + 2y = x^3\sin x$$

$$x^2y'' - 2xy' + 2y = 0$$

let $x=e^t$

$$(m)(m-1) - 2m + 2 = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0 \qquad m=2 \quad m=1$$

$$y_c = C_1e^{2t} + C_2e^t$$

$$= C_1\underbrace{x^2}_{y_1} + C_2\underbrace{x}_{y_2}$$

$$y_p = V_1y_1 + V_2y_2$$

$$V_1 = \int \frac{\begin{vmatrix} 0 & x \\ x\sin x & 1 \end{vmatrix}}{\begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix}}dx = \int \frac{-x^2\sin x}{x^2-2x^2} = \int \sin x = -\cos x$$

$x \quad \sin x$

$1 \quad \cos x$

$0 \quad -\sin x$

$$V_2 = \int \frac{\begin{vmatrix} x^2 & 0 \\ 2x & x\sin x \end{vmatrix}}{\begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix}}dx = \int \frac{x^3\sin x}{-x^2}dx = \int -x\sin x = x\cos x - \sin x$$

$$y = y_c + y_p$$

$$= C_1x^2 - C_2x - \cancel{x}^2\cos x + x^2\cos x - x\sin x$$

$$= C_1x^2 + C_2x - x\sin x$$

2) **(25 points)** Solve the following initial-value problem using Laplace Transform.

$$y'' - 6y' - 7y = \begin{cases} 0 & 0 < t < 1 \\ 56 & t \geq 1 \end{cases}$$

with $y(0) = y'(0) = 0$.

$$s^2 \mathcal{L}\{y(t)\} - s\,y(0) - y'(0) - 6s\mathcal{L}\{y(t)\} + 6y(0) - 7\mathcal{L}\{y(t)\} = \mathcal{L}\{56\,u_1\}$$

$$(s^2 - 6s - 7)\,\mathcal{L}\{y(t)\} = 56\,\frac{e^{-s}}{s}$$

$$\mathcal{L}\{y(t)\} = \frac{56}{s(s-7)(s+1)}e^{-s} = \frac{A e^{-s}}{s} + \frac{B e^{-s}}{s-7} + \frac{C e^{-s}}{s+1}$$

$$A(s-7)(s+1) + B\,s(s+1) + C(s-7)s$$

$$= A(s^2 - 6s - 7) + Bs^2 + Bs + Cs^2 - 7sC = 56$$

$$(A + B + C)s^2 + (-6A + B - 7C)s - 7A = 56$$

$-7A = 56$ $\quad A = -8$

$-6A + B - 7C = 0$ $\quad B = 1$

$A + B + C = 0$ $\quad C = 7$

$$\mathcal{L}\{y(t)\} = \frac{-8e^{-s}}{s} + \frac{e^{-s}}{s-7} + \frac{7e^{-s}}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-8e^{-s}}{s} + \frac{e^{-s}}{s-7} + \frac{7e^{-s}}{s+1}\right\}$$

$$= -8\,u_1 + e^{7(t-1)}u_1 + 7e^{-(t-1)}u_1$$

$$y(t) = \begin{cases} 0 & 0 < t < 1 \\ -8 + e^{7(t-1)} + 7e^{-(t-1)} & t \geq 1 \end{cases}$$

3) **(15 points)** Solve the following linear system

$$x' = 5x + 4y$$
$$y' = -x + y$$

by the matrix method.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0 \qquad (5-\lambda)(1-\lambda) + 4 = 0$$

$$5 - 5\lambda - \lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0 \qquad \lambda_1 = \lambda_2 = 3$$

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = 0 \qquad \begin{aligned} 2\alpha_1 + 4\beta_1 &= 0 \\ -\alpha_1 - 2\beta_1 &= 0 \end{aligned} \rightarrow \beta_1 = -\tfrac{1}{2}\alpha_1$$

$$V_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \begin{aligned} 2\alpha_2 + 4\beta_2 &= -2 \\ -\alpha_2 - 2\beta_2 &= 1 \end{aligned} \rightarrow \beta_2 = -\tfrac{1}{2}\alpha_2 - \tfrac{1}{2}$$

$$V_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{3t} + C_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right] e^{3t}$$

$$x = -2C_1 e^{3t} + C_2(-2t - 3)e^{3t}$$

$$y = C_1 e^{3t} + C_2 (t+1) e^{3t}$$

4) **(20 points)** Solve the following linear system

$$x' = x - 2y$$
$$y' = 4x + 5y \qquad \text{with } x(0) = 1 \text{ and } y(0) = -2$$

by the Laplace transform method.

$$\begin{cases} s\mathcal{L}\{x_{(t)}\} - x_{(0)} = \mathcal{L}\{x_{(t)}\} - 2\mathcal{L}\{y_{(t)}\} \\ s\mathcal{L}\{y_{(t)}\} - y_{(0)} = 4\mathcal{L}\{x_{(t)}\} + 5\mathcal{L}\{y_{(t)}\} \end{cases}$$

$$\begin{cases} (s-1)\mathcal{L}\{x_{(t)}\} + 2\mathcal{L}\{y_{(t)}\} = 1 \\ -4\mathcal{L}\{x_{(t)}\} + (s-5)\mathcal{L}\{y_{(t)}\} = -2 \end{cases}$$

$$\mathcal{L}\{x_{(t)}\} = \frac{\begin{vmatrix} 1 & 2 \\ -2 & s-5 \end{vmatrix}}{\begin{vmatrix} s-1 & 2 \\ -4 & s-5 \end{vmatrix}} = \frac{s-5+4}{s^2-6s+5+8} = \frac{s-1}{s^2-6s+13} = \frac{s-\cancel{3}+2}{(s-3)^2+4}$$

$$= \frac{(s-3)}{(s-3)^2+4} + \frac{2}{(s-3)^2+4} \qquad x_{(t)} = e^{3t}\cos 2t + e^{3t}\sin 2t$$

$$\mathcal{L}\{y_{(t)}\} = \frac{\begin{vmatrix} s-1 & 1 \\ -4 & -2 \end{vmatrix}}{\begin{vmatrix} s-1 & 2 \\ -4 & s-5 \end{vmatrix}} = \frac{-2s+2+4}{(s-3)^2+4} = \frac{-2(s-3)}{(s-3)^2+4}$$

$$y_{(t)} = -2e^{3t}\cos 2t$$

5) **(20 points)**

a) Use the convolution method to find the Laplace inverse of the function $G(s)=\dfrac{2}{s^3(s^2+1)}$.

$$= \frac{2}{s^3} \cdot \frac{1}{s^2+1} = \mathcal{L}\{t^2 * \sin t\}$$

$y \quad \sin y$
$1 \quad \cos y$
$0 \quad \sin y$

mistake

$$\mathcal{L}^{-1}\{G(s)\} = t^2 * \sin t = \int_0^t (t-y)\sin y \, dy$$

$$= \int_0^t t\sin y\,dy + \int_0^t -y\sin y\,dy = -t\cos y\Big|_0^t + y\cos y - \sin y\Big|_0^t$$

$$= -t\cos t + t + t\cos t - \sin t$$

$$= t - \sin t$$

b) Find the Laplace transform of the function $t^2 \sin t$.

$$\mathcal{L}\{t^2 \sin t\} = (-1)^2 \frac{d^2\mathcal{L}\{\sin t\}}{ds^2} = \frac{d^2 \frac{1}{s^2+1}}{dt^2}$$

$$\left(\frac{1}{s^2+1}\right)' = ((s^2+1)^{-1})' = \frac{-2s}{(s^2+1)^2}$$

$$\left(\frac{-2s}{(s^2+1)^2}\right)' = \frac{-2(s^2+1)^2 - 4s(s^2+1)(-2s)}{(s^2+1)^4} = \frac{-2(s^2+1) - 4s(-2s)}{(s^2+1)^3}$$

$$= \frac{-2s^2-2+8s^2}{(s^2+1)^3} = \frac{6s^2-2}{(s^2+1)^3}$$

$$\mathcal{L}\{t^2 \sin t\} = \frac{6s^2-2}{(s^2+1)^3}$$