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1. (12%) Consider the IVP: $e^{2x} y' = \sqrt{y-1}$

$$y(x_0) = y_0$$

Without solving the ODE, determine the largest region of the xy-plane in which the IVP is guaranteed to have a unique solution

$$e^{2x} \frac{dy}{dx} = \sqrt{y-1}$$

$$y' = \frac{\sqrt{y-1}}{e^{2x}} = \frac{(y-1)^{1/2}}{e^{2x}} ;$$

$e^{2x} \neq 0$

$y - 1 \geq 0$

$y \geq 1$

$$y' = \frac{1}{e^{2x}} \cdot \frac{1}{2}(y-1)^{-1/2} = \frac{1}{2e^{2x}\sqrt{y-1}}$$

$e^{2x} \neq 0$

$y - 1 > 0$

$y > 1$ $\Rightarrow$ x & y are continuous on the x-y plane for $y > 1$

$\Rightarrow$ The ~~sater~~ largest ~~interval~~ region is: the x-y plane with $y > 1$

x & $y \in \mathbb{R}$; $y > 1$

2. (20%) Solve the DE: $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y}$.

$$\frac{dy}{dx} = \frac{x+3y}{3x+y}$$

$y =$

dy

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3. (20%) Solve the DE: $\cos x dx + (1+\frac{2}{y})\sin x dy = 0$

$M_y = 1$; $N_x = (1+\frac{2}{y})\cos x$

$\frac{\cos x}{\sin x} dx = -(1+\frac{2}{y}) dy$

$\Rightarrow \ln|\sin x| + C = -y - 2\ln y$

$\Rightarrow y + 2\ln y = \ln|\sin x| + C$

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4. (20%) Solve the DE : $\frac{dy}{dx} = y(xy^3 - 1) = y^4 x - y$

~~let the~~ $\frac{dy}{dx} - y^4 x + y = 0$

$\Rightarrow \frac{dy}{dx} + y = y^4 x$ Bernolli's eq.

let $u = y^{1-4} = y^{-3} = \frac{1}{y^3} \Rightarrow y^3 = \frac{1}{u} \Rightarrow y = \frac{1}{u^{1/3}} = u^{-1/3}$

$\frac{dy}{dx} = -\frac{1}{3} u^{-4/3} \frac{du}{dx}$

replace in DE : $-\frac{1}{3} u^{-4/3} \frac{du}{dx} + u^{-1/3} = u^{-4/3} x$

$\frac{1}{1-4} \frac{du}{dx} + u = x$

$\frac{du}{dx} - 3u = -3x$

$e^{\int -3\,dx} = e^{-3x}$

Multiply in DE:

$e^{-3x} u = \int -3x e^{-3x}\, dx$

$x \quad e^{-3x}$

$1 \quad -3e^{-3x}$

$0 \quad 9e^{-3x}$

$e^{-3x} u = -\int x e^{3x}$

-1

$e^{-3x} u = -3x e^{-3x} - 9e^{-3x} + C$

$u = -3x - 9 + Ce^{3x}$

$\Rightarrow y = (-3x - 9 + Ce^{3x})^{-1/3}$ OPFS

$y=0$ is a singular sol.

3.(15%) Consider the DE: $y'' - 3y' + 2y = 5e^{3x}$

Given that $y_1 = e^x$ is a solution of the associated homogeneous equation, use the method of reduction of order to find another solution y_2 of the homogeneous equation and a particular solution y_p of the given nonhomogeneous equation

$y = u y_1$

$y' = u' y_1 + y_1' u$

$y'' = u'' y_1 + u' y_1' + y_1'' u + u' y_1'$

replace in associated DE :

$u'' e^x + u' e^x + e^x u + u' e^x - 3u' e^x - 3e^x u + 2u e^x = 0$

$u'' e^x + u'(e^x - 3e^x + 2e^x) + e^x u + u' e^x - 3e^x u = 0$

$u'' e^x + e^x u' - 2e^x u = 0$

$e^x \neq 0$

$u'' + u' - 2u = 0$

$u'' e^x + u[e^x - 3e^x + 2e^x] + e^x u' + e^x u' - 3e^x u' = 5e^{3x}$

$u'' e^x - e^x u' = 5e^{3x} \Rightarrow e^x \neq 0 \Rightarrow u'' - u' = 0$

let $w = u'$

$\Rightarrow w' - w = 0 \quad ; e^{\int -1 dx} \quad e^{-x}$

$e^{-x} w = \int 0 \, dx \Rightarrow e^{-x} w = c_1$

$w = c_1 e^x = u'$

$\Rightarrow u = \int c_1 e^x dx = c_1 e^x + c_2$

$y = (c_1 e^x + c_2) e^x = c_1 e^{2x} + c_2 e^x$

y_p ?

$y = y_c + y_p$

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6. (13%) Consider the DE: $x^2y'' - 2xy' + 2y = 0$

a) Given that $y_1 = x$ and $y_2 = x^2$ are solutions of the DE on the interval $(0, \infty)$, show that they form a fundamental set of solutions and write the general solution of the DE on this interval

b) Show why the IVP: $x^2y'' - 2xy' + 2y = 0$
$y(1) = 3$, $y'(1) = 2$
has a unique solution and find that solution

a) $y_1 = x$; $y_2 = x^2$

$\Rightarrow y_2 = f(x)\,y_1$; $f(x) = x$?

since the condition of wronskin are satisfies on $]0;\infty)$

$$\Rightarrow W(y_1; y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

$$y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x^2$$

b) $y' = c_1 + 2c_2 x$

$$\left.\begin{array}{l} 3 = c_1 + c_2 \\ 2 = c_1 + 2c_2 \end{array}\right\} \Rightarrow \text{it has a unique sol.}$$