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Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 235
Ordinary Differential Equations

Exam # 1

Wednesday November 16, 2005

Duration: 55 minutes

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Section: ____________

Instructor: Saleeby

Grade: 95

Directions

1. Write neatly and clearly.
2. Do not use pencils unless for graphing.
3. Show all work.
4. Programmable calculators are not allowed.
5. Turn off your mobile phones.

Please note that you have 6 questions and a total of 7 pages

1) **(15 points)** Find the family of orthogonal trajectories of the family of curves $y^2 = cx$.

$$y^2 = cx$$

$$2y\frac{dy}{dx} = c \qquad \frac{dy}{dx} = \frac{c}{2y} \quad \text{but } c = \frac{y^2}{x} \Rightarrow \frac{dy}{dx} = \frac{y^2}{2xy} = \frac{y}{2x}$$

For orthogonal trajectories: $\frac{dy}{dx} = \frac{-2x}{y}$

this is a separable eq.: $y\,dy = -2x\,dx$

$$\frac{1}{2}y^2 = -x^2 + C_1$$

2) **(15 points)** Find the general solution of the differential equation

$$x^2y'' - x(x+2)y' + (x+2)y = 0$$

given that $y(x) = x$ is a particular solution.

$y_1 = x$ $\qquad$ $P_{(x)} = \dfrac{-x(x+2)}{x^2}$ $\qquad$ $Q_{(x)} = \dfrac{(x+2)}{x^2}$

$$y_2 = y_1 \int \frac{e^{-\int P_{(x)}\,dx}}{y_1^2}\,dx$$

$$y_2 = x \int \frac{e^{\int \frac{(x^2+2x)}{x^2}dx}}{x^2}\,dx$$

$$= x \int \frac{e^{\int (1+2/x)\,dx}}{x^2}\,dx$$

$$= x \int \frac{e^{(x+2\ln|x|)}}{x^2}\,dx$$

$$= x \int \frac{e^x \cdot e^{\ln x^2}}{x^2}\,dx = x \int \frac{e^x}{\cancel{1}}\,dx = xe^x$$

$$y_2 = xe^x$$

general solution is : $y = C_1 x + C_2 xe^x$

3) (20 points) Solve the following initial-value-problem

$y'' - 2y' + y = 2e^x$ with $y(0) = 1$ and $y'(0) = 2$.

$y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$ $\quad m_1 = 1 \quad m_2 = 1$

$y_c = C_1 e^x + C_2 x e^x$

$S \to \{e^x\} \Rightarrow S_1 \to \{xe^x\} \Rightarrow S_2 \to \{x^2 e^x\}$

$\cancel{y_p = Ae^x}$

$\cancel{y'_p = Ae^x}$

$\cancel{y''_p = Ae^x}$

$\cancel{Ae^x - 2Ae^x + Ae^x = 2e^x}$

$\cancel{y_p = Axe^x}$

$\cancel{y'_p = Ae^x + Axe^x}$

$\cancel{y''_p = Ae^x + Ae^x + Axe^x}$

$\cancel{= 2Ae^x + Axe^x}$

$\cancel{2Ae^x + Axe^x - 2Ae^x - 2Axe^x + Axe^x = 2e^x}$

$y_p = Ax^2 e^x$

$y'_p = Ax^2 e^x + 2Axe^x$

$y''_p = Ax^2e^x + 2Axe^x + 2Ae^x + 2Axe^x$

$= Ax^2e^x + 4Axe^x + 2Ae^x$

$Ax^2e^x + 4Axe^x + 2Ae^x - 2Ax^2e^x - 4Axe^x + Ax^2e^x = 2e^x$

$2A = 2$

$A = 1 \qquad y_p = x^2 e^x$

$y = C_1 e^x + C_2 x e^x + x^2 e^x$

$\boxed{1 = C_1}$

$y' = C_1 e^x + C_2 e^x + C_2 x e^x + 2xe^x + x^2 e^x$

$2 = C_1 + C_2$

$\boxed{C_2 = 1}$

$\boxed{y = e^x + xe^x + x^2 e^x}$

4) **(15 points)** Find the general solution of $(2x^2+y)dx+(x^2y-x)dy=0$.

$M \quad dx + N \quad dy = 0$

~~$(M_y - N_x) = 1 - 2xy + 1$~~ ~~$\frac{dy}{dx} = \frac{2x^2+y}{x - x^2y}$~~

$M_y = 1 \qquad N_x = 2xy - 1$ not exact

~~$\frac{1}{N}(M_y - N_x) = \frac{1}{x^2y-x}(-2xy) = \frac{-2y}{xy-1}$~~ ~~$\alpha$~~

~~$\frac{-1}{M}(M_y - N_x) \frac{1}{-2x^2-y}(-2xy)$~~

$M_y - N_x = 2(1-xy)$

$\frac{1}{N}(M_y - N_x) = \frac{1}{x(xy-1)}[2(1-xy)] = \frac{-2}{x}$

$\mu = e^{\int \frac{-2}{x} \cdot dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$

$\left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$

$\int \left(2 + \frac{y}{x^2}\right)dx = 2x - \frac{y}{x} + C_1$

$\int \left(y - \frac{1}{x}\right)dy = \frac{1}{2}y^2 - \frac{y}{x} + C_2$

$g(x,y) = \frac{-y}{x} + \frac{1}{2}y^2 + 2x + C$

$\frac{-y}{x} + \frac{1}{2}y^2 + 2x = C$

5) **(20 points)** Solve the differential equation $x\frac{dy}{dx}+y=y^3$ $(x>0)$.

$\frac{dy}{dX}+\left(\frac{1}{X}\right)y=\frac{1}{X}y^3$ of the form $\frac{dy}{dx}+p(x)y=q(x)y^n$ $n=3$ (Bernoulli

Let $V=y^{1-n}=y^{-2}$

$\frac{dV}{dX}+P(x)V=Q(x)$ where $P(x)=(1-n)p(x)$

$Q(x)=(1-n)q(x)$

$\frac{dV}{dX}-\left(\frac{2}{X}\right)V=\frac{-2}{X}$ ✓ $\frac{dV}{dx}=\frac{-2}{x}+\frac{2}{x}V$

$\mu=e^{-\int P(x)dX}=e^{-2\ln|X|}=e^{\ln X^{-2}}=X^{2}$ X^{-2}

$y=\frac{1}{\mu}\left[\int\mu Q(x)\cdot dX+C\right]$

$y=\frac{1}{X^2}\left[\int\frac{-2}{X^3}\cdot dX+C\right]$

$y=\frac{1}{X^2}\left[\int\frac{1}{X}dX+C\right]$

$y=\frac{1}{X^2}\left[\ln X+C\right]$

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6) **(15 points)** Solve one of the following:

a) $(x+2y-1)dx-(2x+4y+2)dy=0$

b) $\frac{dy}{dx}=(x^2+y)^2-2x+1$ using the transformation $v=x^2+y$.

a) $\frac{dy}{dx}=\frac{x+2y-1}{2x+4y+2}$ $\qquad \frac{2}{1}=\frac{4}{2}=2$

Let $v=x+2y$

$\frac{dv}{dx}=1+2\frac{dy}{dx}=1+2\left(\frac{v-1}{2v+2}\right)$

$=1+\left(\frac{v-1}{v+1}\right)=\frac{v+1+v-1}{v+1}=\frac{2v}{v+1}$

$\frac{v+1}{2v}\,dv=dx$

$\int\left(\frac{1}{2}+\frac{1}{2v}\right)dv=\int dx$

$\frac{1}{2}v+\frac{1}{2}\ln|v|=x+C$

$$\boxed{\frac{1}{2}(x+2y)+\frac{1}{2}\ln|x+2y|=x+C}$$

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1) (15 points) Solve the differential equation $x\frac{dy}{dx} - 3y = -\frac{y^3}{x^4}$, for $x > 0$.

$$\cancel{x\frac{dy}{dx} = -\frac{y^3}{x^4} + 3y = \frac{-y^3+3yx^4}{x^4}}$$

$$\cancel{\frac{dy}{dx} = \frac{-y^3+3yx^4}{x^5}}$$

$\cancel{\text{let } V = \frac{y}{x} \quad y = Vx \quad y' = V'x + V}$

$$\cancel{V'x + V = \frac{-V^3x^3 + 3(Vx)x^4}{x^5}}$$

$$\cancel{V'x + V = \frac{-V^3x^3 + 3Vx^5}{x^5} = \frac{-V^3}{x^2} + 3V}$$

$$\cancel{V'x = \frac{-V^3}{x^2} + 3V - V = \frac{-V^3}{x^2} + 2V}$$

$$\cancel{V'x = \frac{-V^3}{x^2} + 2V}$$

$$\cancel{\frac{dV}{dx}x = \frac{-V^3}{x^2} + 2V}$$

$$\cancel{\frac{dV}{dx} = \frac{-V^3}{x^3} + \frac{2V}{x} = \frac{-V^3 + 2Vx^2}{x^3}}$$

$$\cancel{x^3 dV = (-V^3 + 2Vx^2)dx}$$

$$\cancel{x^3 dV = -V^3 dx + 2Vx^2 dx}$$

sorry for scratch

*)

$$\frac{dy}{dx} - \frac{3}{x}y = -\frac{1}{x^5}y^3 \qquad \text{Bernoulli}$$

$$\boxed{y' - \frac{3}{x}y = \frac{-1}{x^5}y^3} \quad (1)$$

$p(x) = \frac{-3}{x}$ $\qquad n = 3.$

$q(x) = \frac{-1}{x^5}$

let $V = \frac{1}{y^{n-1}} = \frac{1}{y^2}$ $\qquad V' = \frac{-2yy'}{y^4} = -2y^{-3}y'$

divide (1) by y^3:

$$y^{-3}y' - \frac{3}{x}y^{-2} = \frac{-1}{x^5} \Rightarrow -\frac{1}{2}V' - \frac{3}{x}V = \frac{-1}{x^5}$$

15

2) (15 points) Find the family of orthogonal trajectories of the family of curves $y = c\ln x$, for $x > 0$.

orthogonal trajectories:

step 1: $y - c\ln x = 0$

step 2: diff wrt x: $y' - \frac{c}{x} = 0$ 3

step 3: $c/x = y'$ $\quad c = xy'$

We replace in the initial equation

$$y = xy'\ln x = y'x\ln x.$$

3

$$y' = \frac{y}{x\ln x} = f(x,y).$$

$$\frac{dy}{dx} = \frac{-1}{f(x,y)} = \frac{-1}{\frac{y}{x\ln x}} = -\frac{x\ln x}{y}$$

3

$$\frac{dy}{dx} = -\frac{x\ln x}{y}.$$

step 4

$$y\,dy = -x\ln x\,dx$$

$$\int y\,dy = -\int x\ln x\,dx$$

3

$$\frac{y^2}{2} = -\int x\ln x\,dx$$

~~tabular form~~

$$\int x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right] - \int \frac{1}{x}\times\frac{x^2}{2}\,dx.$$

by part:

let $u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$$x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right] - \int\frac{x}{2}\,dx = \left[\frac{x^2}{2}\ln x\right] - \frac{1}{2}\left[\frac{x^2}{2}\right] =$$

$$\frac{y^2}{2} = -\left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c\right] \qquad 2/6 = \left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c\right]$$

15

3) (15 points) Solve the initial-value problem $\frac{dy}{dx} = \frac{(y-3x+5)^2+7}{2}$, with $y(2)=1$.

$$\frac{dy}{dx} = \frac{(y-3x+5)^2+7}{2} \qquad y(2)=1$$

2

let $V = y-3x+5$

1 $V' = y' - 3 \qquad y' = V' + 3$

2 $V' + 3 = \frac{V^2+7}{2}$

$$V' = \frac{V^2+7}{2} - 3 = \frac{V^2+7-6}{2} = \frac{V^2+1}{2} = V'$$

2 $2V' = V^2+1$

$$2\frac{dV}{dx} = V^2+1 \qquad \frac{2}{dx} = \frac{V^2+1}{dV}$$

$$\Rightarrow \frac{dx}{2} = \frac{dV}{V^2+1}$$

$$\int \frac{dV}{V^2+1} = \int \frac{dx}{2} \quad \Rightarrow \int \frac{dV}{V^2+1} = \frac{1}{2}x + \frac{1}{2}C = \frac{1}{2}[x+C]$$

2 $\arctan V = \frac{1}{2}[x+c]$

$$V = \tan\frac{1}{2}[x+c]$$

or $V = y-3x+5$

$$\Rightarrow y-3x+5 = \tan\frac{1}{2}[x+c]$$

2 $y = \tan\frac{1}{2}(x+c) + 3x - 5$

or $y(2)=1 \Rightarrow 1 = \tan\frac{1}{2}(2+c) + \cancel{6} - 5$

2 $\Rightarrow \tan\frac{1}{2}(2+c) = 0$

$$\frac{1}{2}(2+c) = \arc\overset{3/6}{\tan} 0 = 0$$

2 $\Rightarrow \boxed{y = \tan\frac{1}{2}(x-2) + 3x - 5} \qquad 2+c=0 \qquad c=-2$

18

4) (18 points) Solve $\underbrace{(\sin y + x^2 + 2x)}_{M(x,y)}dx + \underbrace{\cos y}_{N(x,y)}dy = 0$. (1)

$\frac{\partial M}{\partial y} = \cos y$

2 not exact.

$\frac{\partial N}{\partial x} = 0$

2 $\frac{1}{N}[M_y - N_x] = \frac{1}{\cos y}[\cos y - 0] = 1 = \underline{\underline{cte}}$.

2 $l(x) = 1$ $\mu(x) = e^{\int dx} = e^x$ which is an integrating factor

multiply (1) by e^x.

2 $e^x[\sin y + x^2 + 2x]dx + e^x \cos y\, dy = 0$.

$e^x \sin y\, dx + e^x x^2 dx + 2e^x x dx + e^x \cos y dy = 0$

4 $d[e^x \sin y] + d(e^x x^2) = d(c)$.

$\Rightarrow$ 4 $d[e^x \sin y + e^x x^2] = d(c)$.

2 $\Rightarrow e^x \sin y + e^x x^2 = C$

$\sin y = \frac{C - e^x x^2}{e^x} \Rightarrow \boxed{y = \arcsin \frac{C - e^x x^2}{e^x}}$ $\to$ answer

20

5) (20 points) Solve the differential equation $x\frac{dy}{dx} - y = x\left(1 - e^{-y/x}\right)$.

$$x\frac{dy}{dx} = x\left[1 - e^{-y/x}\right] + y.$$

3 $$\frac{dy}{dx} = \left[1 - e^{-y/x}\right] + y/x = f(y/x).$$

let $y = Vx$

2 $y' = V'x + V$

$$V'x + V = \left[1 - e^{-V}\right] + V$$

3 $$V'x + \cancel{V} = \left[1 - \frac{1}{e^V}\right] + \cancel{V}$$

$$V'x = \frac{e^V - 1}{e^V}$$

$$\frac{dV}{dx}x = \frac{e^V - 1}{e^V}.$$

6 $$\frac{x}{dx} = \frac{\frac{e^V - 1}{e^V}}{dV} \Rightarrow \frac{dx}{x} = \frac{e^V dV}{e^V - 1}$$

$$\Rightarrow \int \frac{e^V dV}{e^V - 1} = \int \frac{dx}{x}.$$

$$\Rightarrow \int \frac{d(e^V - 1)}{e^V - 1} = \int \frac{dx}{x}.$$

3 $$\Rightarrow \ln|e^V - 1| = \ln x + \ln K = \ln|Kx|.$$

$$\Rightarrow e^V - 1 = Kx$$

$$e^V = Kx + 1$$

or $V = \frac{y}{x}$

$$\Rightarrow e^{y/x} = Kx + 1$$

3 $$y/x = \ln|Kx + 1|$$

$$\Rightarrow \boxed{y = x\ln|Kx + 1|} \rightarrow \text{answer}.$$

17

6) (17 points) Given that $y_1 = x^2$ is a particular solution of the differential equation
$x^3 \dfrac{dy}{dx} - 5x^2 y = -y^2 - 2x^4$, for $x > 0$; find the general solution.

equation !

$$x^3 \frac{dy}{dx} = -y^2 - 2x^4 + 5x^2 y.$$

$$\frac{dy}{dx} = \frac{-1}{x^3} y^2 - 2x + \frac{5}{x} y.$$

2

$$\frac{dy}{dx} - \frac{5}{x} y = \frac{-1}{x^3} y^2 - 2x \quad (1)$$

2

let $y = x^2 + \frac{1}{v}$

$y' = 2x - \frac{v'}{v^2}$

(please turn)

Sorry for scratch (little mistake).

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18

Please note that you have 5 questions and 6 pages

1) **(20 points)** Solve the initial-value problem $y'^2 = y''(y-1)$ with $y(0)=2$, $y'(0)=-1$.

we have $y'' = f(y, y')$.

let $V = y'$ 2

2 $y'' = V' = \frac{dV}{dx} = \frac{dV}{dy}\cdot\frac{dy}{dx} = V\frac{dV}{dy}$

replace:

2 $V^2 = \cancel{V}\frac{dV}{dy}(y-1)$.

$V = \frac{dV}{dy}(y-1)$. for $V \neq 0$ if $V = 0 = y'$ $y = c = 2$.

$\frac{y-1}{dy} = \frac{V}{dV}$ $\frac{dV}{V} = \frac{dy}{y-1}$

$\Rightarrow \int\frac{dy}{y-1} = \int\frac{dV}{V}$

2

$\ln|y-1| = \ln|k_1 V|$

$y-1 = k_1 V$.

$y = k_1 V + 1$

or $V = \frac{dy}{dx}$

$y = k_1\frac{dy}{dx} + 1$

$y-1 = k_1\frac{dy}{dx}$

$k_1 dy = (y-1)\,dx$

4 $\frac{dy}{y-1} = \frac{dx}{k_1}$ $\ln|y-1| = \frac{x}{k_1} + k_2$

$y-1 = e^{\frac{x}{k_1} + k_2}$ 1/6

$y(x) = e^{\left(\frac{x + k_1 k_2}{k_1}\right)} + 1$

8

15

2) (15 points) Find the general solution of the differential equation
$x^4y''+2x^2(x+1)y'+y=0$, for $x>0$ given that $y_1=e^{1/x}$ is a particular solution.

$$y''+\frac{2x^2}{x^4}(x+1)y'+\frac{1}{x^4}y=0$$

standard form:

2 $$y''+\frac{2(x+1)}{x^2}y'+\frac{1}{x^4}y=0$$

$$p(x)=\frac{2(x+1)}{x^2}$$

reduction of order:

2 $$y_2(x)=y_1(x)\int\frac{e^{-\int p(x)dx}}{y_1^2}dx$$

$$y_2(x)=e^{1/x}\int\frac{e^{-\int\frac{2(x+1)}{x^2}dx}}{(e^{1/x})^2}dx$$

$$(e^{1/x})^2=e^{2/x}$$

$$\frac{2x+2}{x^2}=\frac{2}{x}+\frac{2}{x^2}$$

2 $$\int\frac{2(x+1)}{x^2}dx=\int\left(\frac{2}{x}+\frac{2}{x^2}\right)dx$$
$$=2\ln|x|-\frac{2}{x}$$ $x>0$

2 $$e^{-\int\frac{2(x+1)}{x^2}dx}=e^{-(2\ln x-\frac{2}{x})}=e^{\frac{2}{x}-2\ln x}$$

$$y_2(x)=e^{1/x}\int\frac{e^{2/x-2\ln x}}{e^{2/x}}dx=e^{1/x}\int\left(e^{2/x-2\ln x-2/x}\right)dx$$
3 $$=e^{1/x}\int e^{-2\ln x}dx$$
$$=e^{1/x}\int\frac{1}{x^2}dx$$

$$\int\frac{1}{x^2}=-\frac{1}{x}$$ 2 $$y_2(x)=e^{1/x}\times\left(-\frac{1}{x}\right)$$

$$y_2(x)=-\frac{e^{1/x}}{x}$$

2/6

$$y(x)=C_1y_1+C_2y_2$$
$$y(x)=C_1e^{1/x}-C_2\frac{e^{1/x}}{x}$$

15

3) (15 points) ~~Find~~ the general solution of the differential equation $y''' - 3y'' + 2y' = x^2$.

step1: $y''' - 3y'' + 2y' = 0$

$f(r) = r^3 - 3r^2 + 2r = 0$

$r(r^2 - 3r + 2) = 0$

$\boxed{r_1 = 0}$

$r^2 - 3r + 2 = 0$

$a + b + c = 0$ $\quad \boxed{r_2 = 1}$

$\boxed{r_3 = 2}$

$y_c(x) = C_1 e^{0x} + C_2 e^{x} + C_3 e^{2x}$

$y_c(x) = \underset{y_1}{1C_1} + C_2 \underset{y_2}{e^{x}} + C_3 \underset{y_3}{e^{2x}}$

$$w(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$= 1(4e^{3x} - 2e^{3x}) - e^x(4e^{2x}) + e^{2x}(0)$

$= 2e^{3x} - 4e^{3x} = -2e^{3x} \neq 0$

$\boxed{w(y_1, y_2, y_3) = -2e^{3x}}$

$$\begin{pmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x^2 \end{pmatrix}$$

20

4) (20 points) Solve the following differential equation $x^2y'' + 5xy' + 4y = \frac{1}{x}$, for $x > 0$.

step 1: $x^2y'' + 5xy' + 4y = 0$ Cauchy euler's equat.

$f(r) = r(r-1) + 5r + 4 = 0$

$= r^2 - r + 5r + 4 = 0 = r^2 + 4r + 4 = (r+2)^2 = 0$

$f(r) = 0 \Leftrightarrow r^2 + 4r + 4 = 0$

$\Delta = 16 - 16 = 0 \Rightarrow r = \frac{-4}{2} = -2$

(double root).

2

$y_c = C_1 x^{-2} + C_2 x^{-2} \ln x$.

$$y_c = \underbrace{\frac{C_1}{x^2}}_{y_1} + C_2 \underbrace{\frac{\ln x}{x^2}}_{y_2}$$

$$W(y_1, y_2) = \begin{vmatrix} 1/x^2 & \frac{\ln x}{x^2} \\ -2/x^3 & \frac{\frac{1}{x}x^2 - 2x\ln x}{x^4} \end{vmatrix}$$

$$= \begin{vmatrix} 1/x^2 & \frac{\ln x}{x^2} \\ -2/x^3 & \frac{1-2\ln x}{x^3} \end{vmatrix} = \frac{1}{x^2}\left[\frac{1-2\ln x}{x^3}\right] + \frac{2}{x^3}\frac{\ln x}{x^2}$$

2

$$= \frac{1-2\cancel{\ln x}}{x^5} + \frac{2\cancel{\ln x}}{x^5} = \frac{1}{x^5} \neq 0 \quad x>0$$

(y_1, y_2) linearly independent

2

$y_p = V_1 y_1 + V_2 y_2$

2

$$V_1' = \frac{-y_2\, g(x)}{W} = \frac{-\frac{\ln x}{x^2} \times \frac{1}{x^3}}{1/x^5} = -\frac{\ln x}{x^5} \times x^5 = -\ln x$$

1

$V_1 = -\int \ln x\, dx = -(x\ln x - x) = x - x\ln x = V_1$

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5) **(30 points)** Consider the differential equation $(1+x^2)y'' + 2xy' - 2y = 0$.

a) Show that the point $x_0 = 0$ is an ordinary point of the given equation.

b) Find the power series solution for the given equation near $x_0 = 0$.

standard form:

$$y'' + \left(\frac{2x}{1+x^2}\right) y' - \left(\frac{2}{1+x^2}\right) y = 0$$

$\uparrow p(x)$ $\uparrow q(x)$.

$\lim_{x\to 0} p(x) = \lim_{x\to 0} \frac{2x}{1+x^2} = \frac{0}{1+0} = 0$

$\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{-2}{1+x^2} = \frac{-2}{1+0} = -2$

both are analytic at $x_0 = 0$

3

$\Rightarrow x_0 = 0$ is an ordinary pt of the given equation

shortcut method!

$x^0 y^{(2)}$	$x^2 y^{(2)}$	$2x^1 y^{(1)}$	$-2x^0 y^{(0)}$
$k = n-0+2 = n+2$	$k = n$	$k = n$	$k = n$
x^k: $b_{n+2}x^{n+2}$	$b_n x^n$	$b_n x^n$	$b_n x^n$
$x^k)'$: $(n+2)b_{n+2}x^{n+1}$	$n b_n x^{n-1}$	$n b_n x^{n-1}$	X
$x^k)''$: $(n+1)(n+2)b_{n+2}x^n$	$n(n-1)b_n x^{n-2}$	X	X
$(n+1)(n+2)b_{n+2}x^n$	$n(n-1)b_n x^n$	$2nb_n x^n$	$-2b_n x^n$

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$$y(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots b_n x^n + b_{2n} x^{2n} + \cdots b_{n+2} x^{n+2} + \cdots$$

$$= b_0 + b_1 x + b_0 x^2 + 0 + \cdots$$

$$(2) \quad = b_0 \left[1 + \sum_{n=0}^{\infty} x^{n+2} \right] + b_1 (x) \checkmark$$

$$y(x) = b_0 \left[1 + \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n-1} \right] + b_1 x$$ that is

the (1) power series solution near $x_0 = 0$ the ordinary pt

$y_1(x) = b_0 \left[1 + \sum_{n=0}^{\infty} x^{n+2} \right]$ are the 2 solutions rearranged in the

$y_2(x) = b_1 x \checkmark$ upper linear general form.

$$b_0 \left[1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n-1} \right]$$

$$b_{2n+1} = 0$$

$$b_{2n} = (-1)^{n-1} \frac{b_0}{(2n-1)}$$