1) **(10 points)** Let $n = ab$ be the product of positive integers a and b. Prove that either $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

15 2) **(15 points)** Determine whether or not the following argument is valid.

$$\begin{array}{l} p \to r \\ s \to \neg q \\ \underline{q \wedge p} \\ \neg s \wedge r \end{array}$$

$q \wedge p$: true Then q: true and p true.

$p \to r$: true ~~The~~ and we already have p true then r must be true so that $p \to r$ becomes true.

$s \to \neg q$: true and $\neg q$ is false then s must be false so that $s \to \neg q$ becomes true.

S false $\therefore$ $\neg s$: true

r true

Then $\therefore$ $\neg s \wedge r$ is true since r is true & $\neg s$ is true

$\therefore$ The argument is valid

3

3) **(15 points)** Let $A, B, C,$ and D be arbitrary sets. Either prove the given statement is true or provide a counterexample to prove it is false.

$$(A\cup B)\times(C\cup D)=(A\times C)\cup(B\times D)$$

A	B	C	D	A∪B	C∪D
1	1	1	1	1	1
1	1	1	0	1	1
1	1	0	1	1	1
1	1	0	0	1	0
1	0	1	1	1	1
1	0	1	0	1	1
1	0	0	1	1	1
1	0	0	0	1	0
0	1	1	1	1	1
0	1	1	0	1	1
0	1	0	1	1	1
0	1	0	0	1	1
0	0	1	1	0	1
0	0	1	0	0	1
0	0	0	1	0	1
0	0	0	0	0	0

? ?

5

4) **(15 points)** Suppose $f: A\to B$ and $g: B\to C$ are functions. If $g\circ f$ is one-to-one and f is onto, show that g is one-to-one.

gof : one to one ; f onto ; g : one to one?

f : A → B

g : B → C

gof : A → C

let's take $(g\circ f)(a_1) = (g\circ f)(a_2)$

$g(f(a_1)) = g(f(a_2))$

~~for g to be one to one we must have $f(a_1) = f(a_2)$~~

~~since f is onto there must exists two elements a_1 and a_2~~

~~such that $f(a_1) = f(a_2)$~~

~~So g is one to one because $f(a_1) = f(a_2)$~~

$g(a_1) = g(a_2)$

$f(g(a_1)) = f(g(a_2))$

$(f\circ g)a_1 = (f\circ g)(a_1)$

$a_1 = a_2$

$g(a_1) = g(a_2)$

$f(g(a_1)) = f(g(a_2))$

$(f\circ g)a_1 = f a_1 g(a_1)$

a_1

15 **5) (15 points)** Show that

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

whenever n is a positive integer.

for $n=1$; $1^2 = \frac{1(2-1)(2+1)}{3} = \frac{3}{3} = 1$: true ; basis step

Suppose : $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$: True

Let's show it for $(n+1)$: $1^2 + 3^2 + \cdots + (2n-1)^2 + (2n+1)^2$

$$\frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 \overset{?}{=} \frac{(n+1)(2n+1)(2n+3)}{3}$$

$$\cancel{\frac{n(2n-1)(2n+1) + 3(2n+1)^2}{3} = \frac{(2n+1)[n(2n-1) + 3(2n+1)]}{3}}$$

$$\frac{n(2n-1)(2n+1) + 3(2n+1)^2}{3} = \frac{n(4n^2 + 2n - 2n - 1) + 3(4n^2 + 1 + 4n)}{3}$$

$$= \frac{4n^3 - n + 12n^2 + 3 + 12n}{3} = \frac{4n^3 + 11n + 12n^2 + 3}{3}$$

which is equal to $= \frac{(n+1)(2n+1)(2n+3)}{3}$ because this is equal to $=$

$$= \frac{(2n^2 + n + 2n + 1)(2n+3)}{3} = \frac{4n^3 + 6n^2 + 6n^2 + 9n + 2n \ldots}{3}$$

$$= \frac{4n^3 + 12n^2 + 11n + 3}{3} \quad ✓$$

10 **6) (10 points)** Show that the two sets (0,1) and (3,4) have the same cardinality.

$(0,1) \to (3,4)$

$f: x \to x+3$

this function is one to one since $f(x_1) = f(x_2)$

$x_1 + 3 = x_2 + 3$

$x_1 = x_2$

and onto, since $\forall y \in (3,4)\ \exists x \in (0,1) / f(x) = y$

Take $x = y - 3$

$f(y-3) = y - 3 + 3 = y$

It is a bijection, then the two have the same cardinality,

and since $(0,1)$ is uncountable then $(3,4)$ is uncountable

7

7) **(10 points)** Determine whether the following function is one-to-one, onto, or one-to-one correspondence.

$$f : Z \to Z \text{ where } f(n) = \left\lfloor \frac{n}{2} - 1 \right\rfloor.$$

$f(m_1) \overset{?}{=} f(m_2)$

$\left\lfloor \frac{m_1}{2} - 1 \right\rfloor = \left\lfloor \frac{m_2}{2} - 1 \right\rfloor$

$\left\lfloor \frac{m_1}{2} \right\rfloor = \left\lfloor \frac{m_2}{2} \right\rfloor$; $\lfloor m_1 \rfloor = \lfloor m_2 \rfloor$; f is one to one

$m_1 = m_2$

and $\forall y \in \mathbb{Z}, \exists n \in \mathbb{Z} / f(n) = y$

take $n = 2y+2$

$f(2y+2) = \left\lfloor \frac{2y+2}{2} - 1 \right\rfloor$

$= \left\lfloor \frac{2y}{2} + \frac{2}{2} - 1 \right\rfloor = \lfloor y+1-1 \rfloor = \lfloor y \rfloor = y$

then onto.

So it is a one to one correspondance because it is onto and one to one at the same time

10

8) **(10 points)** Show that $[\neg(p \to q)] \to p$ is a tautology without using truth tables.

$[\neg(p \to q)] \to p$

$\Leftrightarrow \neg[\neg(p \to q)] \vee p$

$\Leftrightarrow (p \to q) \vee p$

$\Leftrightarrow (\neg p \vee q) \vee p$

$\Leftrightarrow (\neg p \vee p) \vee q$

$\Leftrightarrow T \vee q$

$\Leftrightarrow T$

It is a tautology