Name: **ID:**

Notre Dame University
MAT 339 (Numerical Analysis)
Exam #1 Fall 2011 50 minutes

1)(25%) a) Prove that the equation $e^x = 3x$ has a unique simple root *r* in the interval $[0,1]$.

b) Apply the bisection method to find r_0, r_1, and r_2; the first three approximations of *r*. Give your answers to the nearest 10^{-3}.

c) Let $e_n = |r_n - r|$ denote the error at the n^{th} step. Find an upper bound for e_n.

d) How many iterations are needed to ensure accuracy to within 10^{-6} of the n^{th} approximation r_n.

2)(25%) On many supercomputers the ratio $\frac{a}{b}$ is computed as $a.\frac{1}{b}$ and $\frac{1}{b}$ is computed by application of Newton's method to the function $f(x) = b - \frac{1}{x}$.

a) Verify that Newton's formula yields $r_{n+1} = r_n(2 - br_n)$ ……… $(*)$ when applied to *f*. Note that no division is involved in $(*)$.

b) Let $e_n = r_n - r$, where *r* is the root of $f(x) = 0$. Prove, in two different ways, that

$$\lim_{n\to\infty}\left|\frac{e_{n+1}}{e_n^2}\right| = |b|.$$

c) Find $\frac{1}{7}$ to three decimals by computing the first three iterates starting with $r_0 = 0.1$.

3)(25%) a) State the Fixed Point Theorem

(b) Verify the four conditions in the Fixed Point Theorem applied to the function $g(x) = \frac{e^{x/2}}{2}$, $0 \le x \le 1$, to conclude that the iteration $r_{n+1} = \frac{e^{r_n/2}}{2}$ converges to the unique root *r* of the equation $e^{x/2} - 2x = 0$ for any choice of r_0 in the interval $[0,1]$.

4)(25%)(a) Write down the Lagrange Interpolating formula with error term that uses data $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_N, f(x_N))$ to approximate $f(x)$ for an arbitrary *x* in $[a,b]$.

(b) For $N \ge 1$, prove that $\sum_{j=0}^{N} Lj(x) = 1$ and $\sum_{j=0}^{N} x_j L_j(x) = x$.

(c) What can be said if $f(x)$ is itself a polynomial of degree $\le N$?