Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 112
Calculus II
Exam # 2
Thursday, May 27, 2007
Duration: 55 minutes

Name: Key

Section: ____________

Instructor: ____________

Grade: ____________

Directions

1. Write neatly and clearly.
2. Do not use pencil except for graphing.
3. Show all work.
4. No calculators are allowed.
5. **No mobile phones whatsoever.**

Please note that you have 8 questions and 7 pages

1) (21 points) Evaluate the following integrals:

a) $\int \frac{\tan(\ln x)}{x}\,dx$.

2 1

Let $u = \ln x \longrightarrow du = \frac{1}{x}dx$

7

1 2

So $\int \frac{\tan(\ln x)\,dx}{x} = \int \tan u\,du = -\ln|\cos u| + C$

$= -\ln|\cos \ln x| + C$.

1

b) $\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}}$

2 1

Let $u = \cos^{-1}x \longrightarrow du = \frac{-dx}{\sqrt{1-x^2}}$

1

7

So $\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}} = -\int e^u\,du = -e^u + C$

2

$= -e^{\cos^{-1}x} + C$.

1

c) $\int x^{2x}(1+\ln x)dx \quad = \int e^{2x\ln x}(1+\ln x)\,dx$.

2

let $u = 2x\ln x \longrightarrow du = \left(\frac{2x}{x} + 2\ln x\right)dx$ 3 1

$= 2(1+\ln x)\,dx$

1

So $\int x^{2x}(1+\ln x)\,dx = \int 2e^{u}\,du = 2e^{u} + C$ 2 1 2

$= 2e^{2x\ln x} + C = 2x^{2x} + C$

1 1

let $u = 2x\ln x \Rightarrow du = 2(\ln x + 1)dx \Rightarrow \frac{du}{2} = (1+\ln x)dx$.

$\int e^{u}\frac{du}{2} = \frac{1}{2}e^{u} + C = \frac{1}{2}e^{2x\ln x} + C = \frac{1}{2}(x)^{2x} + C$.

2) **(8 points)** Solve for y in terms of x
$\ln(y^2-1) - \ln(y+1) = \ln(x^2+1)$

2 2 2 2

$\ln\frac{y^2-1}{y+1} = \ln(x^2+1) \longrightarrow \ln\frac{(y-1)\cancel{(y+1)}}{\cancel{y+1}} = \ln(x^2+1)$

So $y - 1 = x^2 + 1 \longrightarrow y = x^2 + 2$. 2

2 2 2

3) (16 points) Evaluate the following integrals.

a) $\displaystyle\int_0^{\pi/2} \frac{10\cos x\,dx}{1-5\sin x}$ $= I$, Let $u = 1-5\sin x = g(x)$ 2

$\Longrightarrow du = -5\cos x\,dx$ 1

$g(0) = 1-5\sin 0 = 1$. 1

$g(\frac{\pi}{2}) = 1-5\sin\frac{\pi}{2} = 1-5 = -4$ 1

So $I = \displaystyle\int_1^{-4} \frac{-2\,du}{u} = 2\int_{-4}^{1} \frac{du}{u} = 2\ln|u|\Big|_{-4}^{1}$ 2

$= 2[\ln 4 - \ln 1] = 2\ln 4$. 1

b) $\displaystyle\int_1^2 \frac{dx}{x^2-2x+2}$ $= I = \displaystyle\int_1^2 \frac{dx}{x^2-2x+1+1} = \int_1^2 \frac{dx}{(x-1)^2+1}$

2

So let $u = x-1 = g(x) \Rightarrow du = dx$ and 1

$g(1) = 1-1 = 0$, $g(2) = 2-1 = 1$ 2

So $I = \displaystyle\int_0^1 \frac{du}{u^2+1} = \tan^{-1} u\Big|_0^1$ 2

$= \tan^{-1} 1 - \tan^{-1} 0$ 1

$\cancel{= \frac{\pi}{4} - 0 = \frac{\pi}{4}}$

4) **(8 points)** Find the following limit: $\lim_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$.

$$\lim_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}} = \left(e^0+0\right)^{\frac{1}{0}} = 1^{\infty} \text{ indeterminate form.}$$

So we take $\ln\left(e^x+x\right)^{\frac{1}{x}} = \dfrac{\ln\left(e^x+x\right)}{x} \Longrightarrow$

$$\lim_{x\to 0}\ln\left(e^x+x\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{\ln\left(e^x+x\right)}{x} = \frac{\ln 1}{0} = \frac{0}{0} \Longrightarrow$$

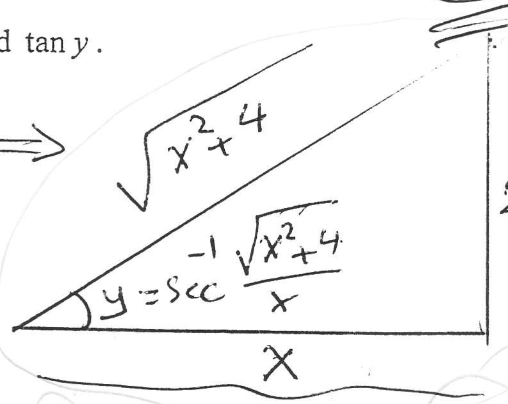
we use H.R. $\Longrightarrow$

$$= \lim_{x\to 0}\frac{\frac{e^x+1}{e^x+x}}{1} = \lim_{x\to 0}\frac{e^x+1}{e^x+0} = 2$$

Therefore $\lim_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}} = e^2$

5) **(8 points)** Given $y=\sec^{-1}\dfrac{\sqrt{x^2+4}}{x}$, find $\sin y$, and $\tan y$.

Since $y=\sec^{-1}\dfrac{\sqrt{x^2+4}}{x} \Longrightarrow$



$\sec y = \dfrac{\sqrt{x^2+4}}{x} \longrightarrow$

$\cos y = \dfrac{x}{\sqrt{x^2+4}}$

$\Rightarrow$ Hence $\sin y = \dfrac{2}{\sqrt{x^2+4}}$

and $\tan y = \dfrac{2}{x}$

6) (15 points) Find the absolute maximum and minimum of the function $y = f(x) = 20x - 10x\ln x$ on $\left[\frac{1}{e}, e^2\right]$.

(2) $f'(x) = 20 - [\frac{10x}{x} + 10\ln x] = 20 - 10 - 10\ln x = 10 - 10\ln x$

which is defined forall $x \in [\frac{1}{e}, e^2]$.

(2) $f'(x) = 0 \rightarrow 10 - 10\ln x = 0 \rightarrow \ln x = 1 \rightarrow x = e$

which is between $\frac{1}{e}$ and e^2.

(2) $f(\frac{1}{e}) = 20(\frac{1}{e}) - 10(\frac{1}{e})\ln\frac{1}{e}$

$= \frac{20}{e} + \frac{10}{e}\ln e = \frac{30}{e}$

(2) $f(e) = 20e - 10e\ln e = 20e - 10e = 10e$

(2) $f(e^2) = 20e^2 - 10e^2\ln e^2 = 20e^2 - 2(10e^2\ln e)$

$= 20e^2 - 20e^2 = 0$.

(2.5) So $f(x)$ has absolute max. value $= 10e$ at $x = e$

Abs max (2)

(2.5) and $f(x)$ has absolute min-value $= 0$ at $x = e^2$.

Abs min (2)

7) (12 points) Find the derivative of the following functions:

a) $y = x^{\sin x}$ $\qquad x > 0$

$$y = x^{\sin x} = e^{\sin x \ln x} \Longrightarrow$$

$$y' = e^{\sin x \ln x}\left[\frac{\sin x}{x} + \cos x \ln x\right]$$

6

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ln x\right].$$

6 pt

b) $e^{2x} = \ln(x+3y)$

$$2e^{2x} = \frac{1}{x+3y}\left[1 + 3\frac{dy}{dx}\right] = \frac{1}{x+3y} + 3\frac{dy}{dx}\frac{1}{x+3y}$$

6

$$\left[2e^{2x} - \frac{1}{x+3y}\right] = \frac{dy}{dx}\frac{3}{x+3y} \longrightarrow$$

$$\frac{dy}{dx} = \left(2e^{2x} - \frac{1}{x+3y}\right)\cdot\frac{x+3y}{3}$$

6 pt

8) (12 points) Given $y = f(x) = \log_{\frac{1}{3}} x$ Domain $x > 0$, Rang $y \in \mathbb{R}$.

a) Find $f^{-1}(x)$. What is the Domain and range of $f^{-1}(x)$?

3 — $f^{-1}(x) = \left(\frac{1}{3}\right)^x$ 3

1.5 1.5 D: all $x \in \mathbb{R}$

1.5 1.5 R: all $y > 0$

$y = \log_{\frac{1}{3}} x = \frac{\ln x}{\ln \frac{1}{3}} \rightarrow$

$y \ln \frac{1}{3} = \ln x \rightarrow$

$\ln\left(\frac{1}{3}\right)^y = \ln x \rightarrow$

$x = \left(\frac{1}{3}\right)^y \Rightarrow$

$y = f^{-1}(x) = \left(\frac{1}{3}\right)^x$.

b) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ together.

6 6