Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 112

Calculus II

Exam # 2

Thursday June 1st, 2006

Duration: 60 minutes

Name: ____________________

Section: ____________________

Instructor: ____________________

Key

Grade: ____________________

Directions

1. Write neatly and clearly.
2. Do not use pencils unless for graphing.
3. Show all work.
4. No calculators are allowed.
5. Turn off your mobile phone.

Please note that you have 6 questions and 7 pages

1) (25 points) Evaluate each of the following integrals:

a) $\int \frac{\sin\theta \, d\theta}{1-4\cos\theta} = \frac{1}{4}\int \frac{du}{u} \; (2)\text{pts} = \frac{1}{4}\ln|u| + C \; (2)\text{pts} = \frac{1}{4}\ln|1-4\cos\theta| + C. \; (1)\text{pt}$

Let $u = 1-4\cos\theta \Rightarrow du = -4(-\sin\theta)\,d\theta \Rightarrow du = 4\sin\theta\,d\theta$

$\Rightarrow \frac{du}{4} = \sin\theta\,d\theta$

b) $\int_1^{e^2} \frac{3^{\ln t}\,dt}{t} = \int_0^2 3^u\,du \; (2)\text{pts} = \frac{3^u}{\ln 3}\Big]_0^2 \; (2)\text{pts} = \frac{3^2 - 3^0}{\ln 3} = \frac{8}{\ln 3} \; (1)\text{pt}$

Let $u = \ln t = g(t) \Rightarrow du = \frac{1}{t}dt$, $g(1) = 0$, $g(e^2) = \ln e^2 = 2$

c) $\int \sinh(\ln x)dx = \int \frac{e^{\ln x} - e^{-\ln x}}{2}dx = \int \frac{x - e^{\ln\frac{1}{x}}}{2}dx$

$= \int \frac{x - \frac{1}{x}}{2}dx = \frac{1}{2}\left[\frac{x^2}{2} - \ln|x|\right] + C.$

(3)pts (2)pts

d) $\int \frac{dx}{\sqrt{3x^2-6x-9}} = \int \frac{dx}{\sqrt{3x^2-6x+3-12}} = \int \frac{dx}{\sqrt{3(x^2-2x+1)-12}}$

$$= \int \frac{dx}{\sqrt{3(x-1)^2-12}} = \int \frac{dx}{\sqrt{3}\sqrt{(x-1)^2-4}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{4\left(\frac{x-1}{2}\right)^2-1}} = \frac{1}{2\sqrt{3}} \int \frac{dx}{\sqrt{\left(\frac{x-1}{2}\right)^2-1}} \quad 3\ pts$$

$$= \frac{1}{2\sqrt{3}} \frac{\cosh^{-1}\left(\frac{x-1}{2}\right)}{\frac{1}{2}} + C$$

$$= \frac{1}{\sqrt{3}} \cosh^{-1}\left(\frac{x-1}{2}\right) + C. \quad 2\ pts$$

e) $\int \frac{x^2dx}{x^2+4} = \int \frac{x^2+4-4}{x^2+4} dx = \int dx - 4\int \frac{dx}{x^2+4}. \quad 1\ pt$

$$= x - 4\int \frac{dx}{4\left(\left(\frac{x}{2}\right)^2+1\right)} = x - \int \frac{dx}{1+\left(\frac{x}{2}\right)^2} \quad 2\ pts$$

$$= x - \frac{\tan^{-1}\left(\frac{x}{2}\right)}{\frac{1}{2}} + C = x - 2\tan^{-1}\left(\frac{x}{2}\right) + C. \quad 2\ pts.$$

2) **(15 points)** Find the length of the curve $y = \frac{e^x}{4} + e^{-x}$ from $x = 0$ to $x = 1$.

$\frac{dy}{dx} = \frac{e^x}{4} - e^{-x}$ from $x=0$ to $x=1$. is cont. 2 pt.

$\therefore L = \int_0^1 \sqrt{1 + \left(\frac{e^x}{4} - e^{-x}\right)^2}\, dx$ 4 pt.

$= \int_0^1 \sqrt{1 + \frac{e^{2x}}{16} - \frac{1}{2} + e^{-2x}}\, dx$

$= \int_0^1 \sqrt{\frac{e^{2x}}{16} + \frac{1}{2} + e^{-2x}}\, dx$

$= \int_0^1 \sqrt{\left(\frac{e^x}{4} + e^{-x}\right)^{\cancel{2}}}\, dx = \int_0^1 \frac{e^x}{4} + e^{-x}\, dx$ 4 pts

$= \frac{e^x}{4} - e^{-x} \Big]_0^1$ 3 pts $= \frac{e}{4} - e^{-1} - \left(\frac{1}{4} - 1\right)$

$= \frac{e}{4} - \frac{1}{e} + \frac{3}{4} = \frac{e^2 - 4}{4e} + \frac{3}{4} = \frac{3.389}{4e} + 0.75$

$\simeq 1.0616$ 2 pts

3) **(15 points)** Find the absolute maximum and absolute minimum values of the function $y = \dfrac{\ln x}{\sqrt{x}}$ on $[2,16]$.

$$y' = \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{x} = \frac{\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x\sqrt{x}} = 0$$

$\Rightarrow \ln x = 2 \Rightarrow x = e^2$

$+ \phi -$

5 pts

* critical pt $\left(e^2, f(e^2) = \frac{2}{e} \cong 0.7357\right)$. 2 pts

* end pts $\left(2, f(2) = \frac{\ln 2}{\sqrt{2}} \cong 0.490\right)$ 2 pts

$\left(16, f(16) = \frac{4\ln 2}{4} = \ln 2 \cong 0.69\right)$ 2 pts

Abs max $= 0.735 = \frac{2}{e}$ at $x = e^2$. 2 pts

Abs min $= 0.49$ at $x = 2$. 2 pts

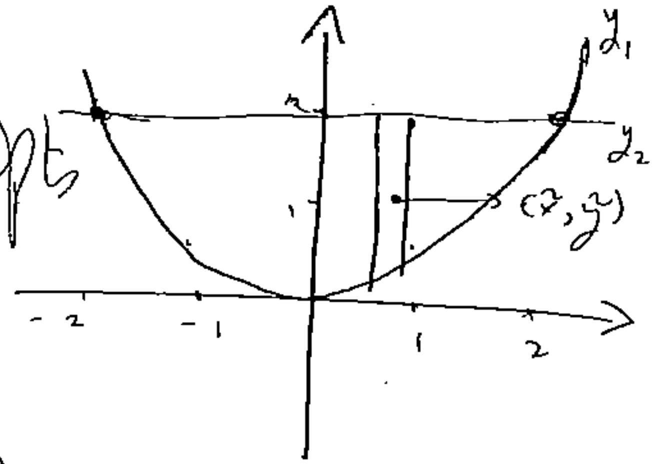
4) **(21 points)** Find the center of mass of a thin plate of constant density covering the region enclosed by the parabola $y=\frac{x^2}{2}$, and the line $y=2$.

* $y = \frac{x^2}{2}$

x	-2	-1	0	1	2
y	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2

<u>Note:</u> $y=y \Rightarrow \frac{x^2}{2}=2 \Rightarrow x^2=4$
$\Rightarrow x = \pm 2$.

3 pts



2 pts: $\therefore$ C.M $(\bar{x},\bar{y}) = (0,\bar{y})$.
$\bar{x}=0$ because y-axis is an axis of sym.

to Find $\bar{y}$ consider vertical thin strips.

$$\boxed{\therefore \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} = \frac{M_x}{M}.}$$ 2 pts

2 pts: $\tilde{y} = \frac{y_1+y_2}{2} = \frac{\frac{x^2}{2}+2}{2} = \frac{x^2+4}{4}$; $dm=\delta dA = \delta(y_2-y_1)\,dx$

$dm = \delta\left(2-\frac{x^2}{2}\right)dx$
$=\delta\left(\frac{4-x^2}{2}\right)dx$ 2 pts

$\therefore M_x = \int_{-2}^{2} \frac{4+x^2}{4}\left(\frac{\delta}{2}(4-x^2)\,dx\right)$

$= \frac{\delta}{8}\int_{-2}^{2}(16-x^4)\,dx$ 2 pts $= \frac{\delta}{8}\left[16x-\frac{x^5}{5}\right]_{-2}^{2}$

$=\frac{\delta}{8}\left[32-\frac{32}{5}+32-\frac{32}{5}\right] = \frac{\delta}{8}\left[64-\frac{64}{5}\right] = \delta\left[8-\frac{8}{5}\right] = \frac{32\delta}{5}$ 2 pts

$M=\int_{-2}^{2} dm = \frac{\delta}{2}\int_{-2}^{2}(4-x^2)\,dx$ 2 pts $= \frac{\delta}{2}\left[4x-\frac{x^3}{3}\right]_{-2}^{2} = \frac{\delta}{2}\left[8-\frac{8}{3}+8-\frac{8}{3}\right]$

$=\frac{\delta}{2}\left[16-\frac{16}{3}\right] = \delta\left[8-\frac{8}{3}\right] = \delta\left[\frac{16}{3}\right]$ 2 pts

$\therefore \bar{y} = \frac{32\delta}{5}\Big/\frac{16\delta}{3} = \frac{32}{5}\times\frac{3}{16} = \frac{6}{5} = 1.2$ $\therefore$ C.M $(0, 1.2)$ 2 pts

5) (12 points)

a) Find the derivative of $y = x^{\sinh x}$. $= e^{(\sinh x)(\ln x)}$

$$\therefore \frac{dy}{dx} = e^{(\sinh x).(\ln x)} \cdot \left((\cosh(x)) (\ln x) + \frac{\sinh x}{x} \right)$$

$$\frac{dy}{dx} = x^{\sinh x} \left((\cosh x).(\ln x) + \frac{\sinh x}{x} \right)$$

6 pts

b) Use logarithmic differentiation to find the derivative of $y = \frac{(1-x)^2(1+x)^{3/4}}{(x^2-x)^3}$.

$$\ln y = 2 \ln|1-x| + \frac{3}{4} \ln|1+x| - 3 \ln(x^2-x)$$

2 pts

$$\Rightarrow \frac{y'}{y} = \frac{2}{1-x}(-1) + \frac{3}{4}\frac{1}{1+x} - \frac{3}{x^2-x}(2x-1)$$

$$\Rightarrow y' = y\left(\frac{-2}{1-x} + \frac{3}{4(1+x)} - \frac{6x-3}{x^2-x} \right)$$

3 pts

$$y' = \frac{(1-x)^2(1+x)^{3/4}}{(x^2-x)^3} \left(\frac{-2}{1-x} + \frac{3}{4(1+x)} - \frac{6x-3}{x^2-x} \right).$$

1 pt

6) **(12 points)** Find the area of the region enclosed by the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

Sol.

$2x > x$ for $x \in [1,5]$

$\Rightarrow \ln 2x > \ln x$ for $x \in [1,5]$

$$\Rightarrow A = \int_1^5 \ln 2x - \ln x \, dx \quad (6) \text{ pts}$$

$$= \int_1^5 \ln\left(\frac{2x}{x}\right) dx = \int_1^5 \ln 2 \, dx \quad (4) \text{ pts}$$

$$= \ln 2 \, [x]_1^5 = 4 \ln 2. \quad (2) \text{ pts}$$