$(\cos^{-1})' = \frac{-1}{\sqrt{1-x^2}}$

$(\tan^{-1})' = \frac{1}{1+x^2}$

$(\cot^{-1})' = \frac{-1}{1+x^2}$

$(\sec^{-1})' = \frac{1}{|x|\sqrt{x^2-1}}$

$(\csc^{-1})' = \frac{-1}{|x|\sqrt{x^2-1}}$

$\sinh^{-1}{}' = \frac{1}{\sqrt{1+x^2}}$

$\cosh^{-1}{}' = \frac{1}{\sqrt{x^2-1}}$

$\tanh^{-1}{}' = \coth^{-1}{}' = \frac{1}{1-x^2}$

$\operatorname{sech}^{-1}{}' = \frac{1}{|x|\sqrt{1-x^2}}$ $\qquad \operatorname{csch}^{-1}{}' = \frac{1}{|x|\sqrt{x^2+1}}$

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Please note that you have 6 questions and 6 pages

1) **(28 points)** Evaluate the following integrals:

a) $\displaystyle\int_0^{\pi/2} \frac{\sin x\,dx}{1+\cos^2 x}$

$\boxed{\begin{array}{l} \text{Let } u = \cos x \\ du = -\sin x\,dx \end{array}}$

~~$\cot^{-1}(0) = \alpha$~~

~~$\frac{\cos^{-1}(0)}{\sin^{-1}(0)} = \alpha$~~

~~$\frac{\pi/2}{0} = \alpha$~~

~~$\alpha = 0$~~

(~~because the boundaries became~~ $\int_1^0$).

$\cot^{-1}(-1) = \beta$

$\frac{\cos^{-1}(1)}{\sin^{-1}(1)} = \beta$

$\underbrace{\cot^{-1}\left(\frac{\pi}{2}\right)}_{\alpha} - \underbrace{\cot^{-1}(0)}_{\beta}$

$\cancel{I = -\left[\cot^{-1}\left(\cos\frac{\pi}{2}\right) - \cot^{-1}(\cos 0)\right]}$

$\cancel{= -\left[\underbrace{\cot^{-1}(0)}_{\alpha} - \underbrace{\cot^{-1}(1)}_{\beta}\right]}$

b) $\displaystyle\int \sinh(\ln x)\,dx$ for $x > 0$.

we know that $\sinh(x) = \frac{e^x - e^{-x}}{2}$

$I = \displaystyle\int \frac{e^{\ln x} - e^{-\ln x}}{2}\,dx$

$I = \displaystyle\int \frac{x - x^{-1}}{2}\,dx = \int \frac{x}{2} - \frac{1}{2x}\,dx = \int \frac{x^2-1}{2x}\,dx$

$= \displaystyle\int \frac{1}{2}\left(\frac{x^2-1}{x}\right) = \frac{1}{2}\int x - \frac{1}{x} = \frac{1}{2}\left(\frac{x^2}{2} - \ln|x|\right)$

$I = \frac{x^2}{4} - \frac{\ln x}{2} + c.$ $(x>0)$.

$-2x = 2x\,b$

$-x = x\,b$

$\boxed{-1 = b}$

$(x-1)^2 = x^2+1-2x$

$-12 + 3x^2 + 3 - 6x$

c) $\displaystyle\int \frac{dx}{\sqrt{3x^2-6x-9}}$

Let $P(x) = 3x^2-6x-9$.

$= 3(x^2-2x-3)$.

$= 3[(x-1)^2-1-3]$

$= 3((x-1)^2-4)$.

$$\int \frac{dx}{\sqrt{3((x-1)^2-4)}} = \int \frac{dx}{\sqrt{3(x-1)^2-12}} = \int \frac{dx}{\sqrt{3}\times\sqrt{(x-1)^2-4}}$$

$$I = \int \frac{dx}{\frac{\sqrt{3}}{4}\sqrt{\left(\frac{x-1}{2}\right)^2-1}} = \left(\frac{\sqrt{3}}{4}\right)^{-1}\int \frac{dx}{\left(\frac{x-1}{2}\right)^2-1} = \frac{4}{\sqrt{3}}\times 2\cosh^{-1}\left(\frac{x-1}{2}\right)+c$$

$$= \frac{8}{\sqrt{3}}\cosh^{-1}\left(\frac{x-1}{2}\right)+c.$$

d) i) $\int 2x^{2x}(1+\ln x)\,dx$ for $x>0$ or

ii) $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x-8}}$ for $x>4$.

Note that, if you solve (i) you get 5 points as bonus.

$$\int 2x^{2x}(1+\ln x)\,dx = \int 2e^{2x\ln x}(1+\ln x)\,dx.$$

Let

$2(x\ln x) = u$.

$2\left(\ln x + \frac{1}{x}\times x\right)dx = du$

$2(\ln x+1) = du$.

So $I = \int e^u\,du$.

$= e^u + c = x^{2x}+c$.

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2) **(9 points)** Find the mean value of the function $f(x)=\frac{1}{e^{-x}+1}$ from $x=0$ to $x=\ln 2$.

Mean value of $\int_0^{\ln 2} \frac{1}{e^{-x}+1}\,dx$

is $= \frac{1}{\ln 2 - 0}\int_0^{\ln 2}\frac{1}{e^{-x}+1}\,dx$.

average $f = \frac{1}{\ln 2}\int_0^{\ln 2}\frac{1}{e^{-x}+1}\,dx$.

$= \frac{1}{\ln 2}\int_0^{\ln 2}\frac{1}{e^{-x}(1+e^{x})} = \frac{1}{\ln 2}\int_0^{\ln 2}\frac{e^{x}}{1+e^{x}}\,dx$.

$= \frac{1}{\ln 2}\int_0^{\ln 2}\frac{du}{u}$

Let $u = 1+e^{x}$

$du = e^{x}\,dx$.

$= \frac{1}{\ln 2}\times \ln|u|\Big|_0^{\ln 2}$

The boundaries become: $\int_2^{1+\ln 2}$. $>0 \Rightarrow$ possible

$= \frac{1}{\ln 2}\times\left(\ln(1+\ln 2) - \ln(2)\right)$

$= \frac{1}{\ln 2}\times\left(\ln(1+\ln 2)\right) - 1$.

$= \frac{\ln(1+\ln 2)}{\ln 2} - 1$.

3) (17 points) Find $\frac{dy}{dx}$ for the following.

a) $y = (1-x)^2(1+x)^{3/4}/(x^2-x)^3$

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$$y = \frac{(1-x)^2(1+x)^{3/4}}{(x^2-x)^3}$$

$$\ln y = \ln(1-x)^2 + \ln(1+x)^{3/4} - \ln(x^2-x)^3.$$

$$\ln y = 2\ln(1-x) + \frac{3}{4}\ln(1+x) - 3\ln(x^2-x).$$

$$\frac{y'}{y} = 2\times\left(\frac{-1}{1-x}\right) + \frac{3}{4}\left(\frac{1}{1+x}\right) - 3\left(\frac{2x-1}{x^2-x}\right).$$

$$y' = \left[\frac{-2}{1-x} + \frac{3}{4(1+x)} - 3\left(\frac{2x-1}{x^2-x}\right)\right]\times\frac{(1-x)^2(1+x)^{3/4}}{(x^2-x)^3}$$

b) $y = 2^{\sin x}$ for $\sin x > 0$

$$y = 2^{\sin x}.$$

$$\ln y = \ln 2^{\sin x}$$

$$\ln y = \sin x \ln 2.$$

$$\frac{y'}{y} = \cos x . \ln 2$$

$$y' = \cos x \times \ln 2 \times 2^{\sin x}.$$

c) $y = \sin^{-1}\sqrt{1-u^2}$ for $0 < u < 1$

$$(\sin^{-1})' = \frac{(x')}{\sqrt{1-(x)^2}}.$$

$$y = \sin^{-1}\sqrt{1-u^2}.$$

$$= \frac{-u/\sqrt{1-u^2}}{\sqrt{1-(\sqrt{1-u^2})^2}}$$

$$(\sqrt{1-u^2})' = \frac{-2u}{2\sqrt{1-u^2}} = -\frac{u}{\sqrt{1-u^2}}.$$

$$= \frac{-u}{\sqrt{1-u^2}}\times\frac{1}{\sqrt{1-(1-u^2)}}$$

$1 - 1 + u^2$

$$= \frac{-u}{\sqrt{1-u^2}}\times\frac{1}{\sqrt{u^2}}$$

$$= \frac{-\cancel{u}}{\sqrt{1-u^2}}\times\frac{1}{\cancel{u}}$$

$$= -\frac{1}{\sqrt{1-u^2}}.$$

4) (5 points) Solve for x: $\ln(x-1) = y + \ln x$

$\ln(x-1) = y + \ln x$

$y = \ln(x-1) - \ln x$

$y = \ln\left(\frac{x-1}{x}\right)$

5) (18 points) Find the length of the curve $y = \ln x - \frac{x^2}{8}$ from $x=1$, and $x=2$.

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$L = \int_1^2 \sqrt{1+y'^2}\, dx$

$y' = \frac{1}{x} - \frac{1}{8} \times 2x$

$= \frac{1}{x} - \frac{1}{4}x$

$2 \times \frac{1}{x} \times \left(-\frac{1}{4}x\right)$

$-\frac{1}{2}$

$L = \int_1^2 \sqrt{1+\left(\frac{1}{x} - \frac{1}{4}x\right)^2}\, dx$

$= \int_1^2 \sqrt{1+\left(\frac{1}{x}\right)^2 + \left(\frac{1}{4}x\right)^2 - \frac{1}{2}}\, dx$

$= \int_1^2 \sqrt{\frac{1}{2} + \left(\frac{1}{x}\right)^2 + \left(\frac{1}{4}x\right)^2}\, dx$

$= \int_1^2 \sqrt{\left(\frac{1}{x} + \frac{1}{4}x\right)^2}\, dx$

$= \int_1^2 \frac{1}{x} + \frac{1}{4}x\, dx \qquad x > 0$

$= \ln|x| + \frac{1}{4}\frac{x^2}{2}\Big|_1^2$

$= \ln 2 + \frac{1}{4}\frac{(\ln 2)^2}{2} - \Big[\underbrace{\ln(1) + \frac{1}{4}\frac{(\ln 1)^2}{2}}_{0}\Big]$

$= \ln 2 + \frac{1}{4}\frac{(\ln 2)^2}{2}$

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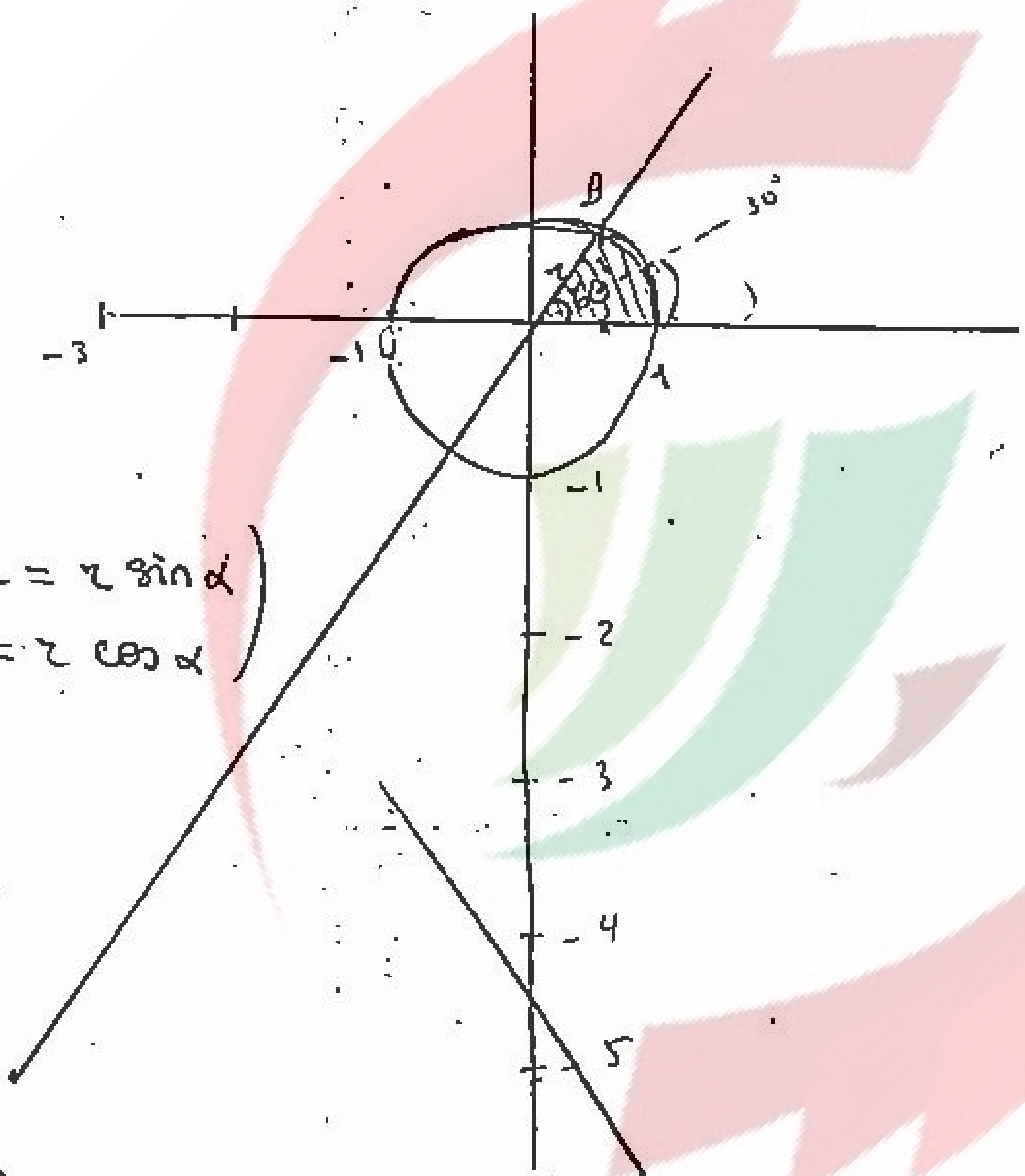
6) (23 points) Find the center of mass of a thin plate of constant density enclosed between $y=\sqrt{(1-x^2)}$ and the x-axis and the line $y=\sqrt{3}\,x$.

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

equatiO of a circle $(0;0)$ center of the circle and $R^2 = 1 \Rightarrow R = 1$ of the cercle (radius).



x	1	2
y	$\sqrt{3}$	$2\sqrt{3}$

$\left(\begin{aligned} x &= r \sin\alpha \\ y &= r\cos\alpha \end{aligned}\right)$

$y = \sqrt{3}\,x$

$y = (\tan\alpha)\,x$

what's α?

$\alpha = \tan^{-1}(\sqrt{3})$

$\alpha = 60°$

$M = \int_0^1 \sigma \, dA$

$= \sigma A \Big|_0^1$

the area is: $360° \longrightarrow \pi r^2$

$60° \longrightarrow ?$

$\frac{\pi}{6}$

$$\boxed{M = \frac{\sigma\pi}{6}}$$

symetry with respect to $y = (\tan 30)\,x$

$\bar{y} = (\tan 30)\,\bar{x}$

$\frac{M_x}{M} = (\tan 30)\frac{M_y}{M}$

$M_x = \frac{\sqrt{3}}{2} M_y$

$$M_x = \int_0^{\sin 60} \frac{\sigma}{2}\big(\underbrace{y_{up}^2 - y_{lo}^2}_{\text{of the line}}\big)\,dx + \int_{\sin 60}^{1} \frac{\sigma}{2}\big(\underbrace{y_{up}^2 - y_{lo}^2}_{\text{of circle}}\big)\,dx$$

$$= \int_0^{\sin 60} \frac{\sigma}{2}(3x^2 - 0)\,dx + \int_{\sin 60}^{1}\frac{\sigma}{2}(1 - x^2 - 0)\,dx.$$

$$= \int_0^{\sin 60} \frac{\sigma}{2}(3x^2)\,dx + \int_{\sin 60}^{1}\frac{\sigma}{2}(1 - x^2)\,dx$$

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$$= \frac{\sigma}{2} \times \frac{3}{3} x^3 \Big|_0^{\sin 60} + \frac{\sigma}{2} \times \left(x - \frac{x^3}{3}\right)\Big|_{\sin 60}^{1} =$$

$$= \frac{\sigma}{2}(\sin 60)^3 + \frac{\sigma}{2}\left[\left(1 - \frac{1}{3}\right) - \left(\sin 60 - \frac{(\sin 60)^3}{3}\right)\right] = \frac{\sigma}{2}\times(0.5)^3 + \frac{\sigma}{2}\left(1 - \frac{1}{3} - 0.5 + \frac{(0.5)^3}{3}\right.$$