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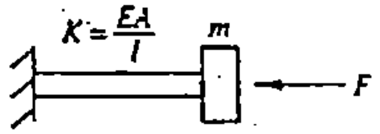
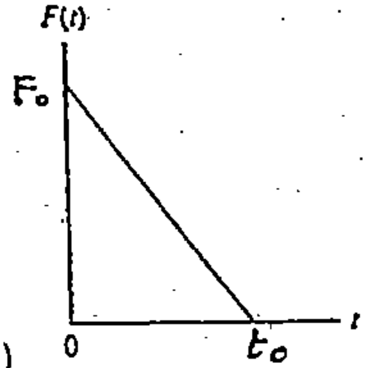
MEN 330
EXAM 2
FALL 2003

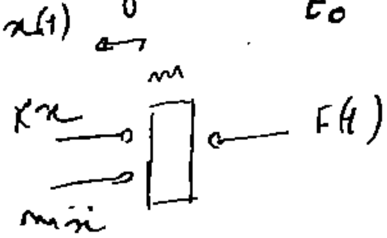
1) (15 pts) Use the Laplace transform to determine the response of the beam in the axial direction if the graph of the input function $F(t)$ is as shown below. Assume zero initial conditions.





$$F(t) = -\frac{F_0}{t_0}t + F_0$$

eq. of the st. line.



$\sum F_x = 0$

$$m\ddot{x} + Kx = F(t)$$

$$m\ddot{x} + Kx = \left\{ -\frac{F_0}{t_0}t + F_0 \right.$$

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$$F(t) = \begin{cases} -\frac{F_0}{t_0}t, & 0 < t < t_0 \\ 0; & t > t_0 \end{cases}$$

$$(ms^2 + K)X(s) = -\frac{F_0}{t_0} \times \frac{1}{s^2} + \frac{F_0}{s}$$

$$X(s) = \left(\frac{1}{ms^2+K}\right)\left(-\frac{F_0}{t_0} \times \frac{1}{s^2} + \frac{F_0}{s}\right)$$

$$F(t) = -\frac{F_0}{t_0}t\,u(t-0) + \frac{F_0}{t_0}t\,u(t-t_0)$$

$$X(s) = F_0\left(-\frac{1}{t_0(s^2)(ms^2+K)} + \frac{1}{s(ms^2+K)}\right)$$

$$x(t) = \frac{F_0}{m t_0}\mathcal{L}^{-1}\left[-\frac{1}{s^2(s^2+\omega_n^2)} + \frac{t_0}{s(s^2+\omega_n^2)}\right]$$

$(t-t_0)u(t-t_0) + t_0 u(t-t_0)$

$$\frac{1}{s^2(s^2+\omega_n^2)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+\omega_n^2}$$

$$\Rightarrow As^3 + As\omega_n^2 + Bs^2 + B\omega_n^2 + Cs^3 + Ds^2 = 1$$

$$\Rightarrow \boxed{B = \frac{1}{\omega_n^2}} \; ; \quad A\omega_n^2 = 0 \Rightarrow \boxed{A = 0} \; ; \quad A + C = 0 \Rightarrow \boxed{C = 0}$$

$$D = -\frac{1}{\omega_n^2}$$

$$\Rightarrow -\frac{1}{s^2(s^2+\omega_n^2)} = \frac{-1/\omega_n^2}{s^2} + \frac{1/\omega_n^2}{s^2+\omega_n^2}$$

$$\frac{t_0}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+D}{s^2+\omega_n^2}$$

X

$$\Rightarrow As^2 + A\omega_n^2 + Bs^2 + Ds = t_0$$

$$\Rightarrow A + B = 0 \Rightarrow \boxed{B = -\frac{t_0}{\omega_n^2}} \qquad \boxed{D = 0}$$

$$\boxed{A = \frac{t_0}{\omega_n^2}}$$

$$\frac{t_0}{(s^2+\omega_n^2)} = \frac{\frac{t_0}{\omega_n^2}}{s} + \frac{-\frac{t_0}{\omega_n^2}}{s^2+\omega_n^2}$$

$$x(t) = \frac{f_0}{mt_0}\mathcal{L}^{-1}\left[\frac{-\frac{1}{\omega_n^2}}{s^2} + \frac{\frac{1}{\omega_n^2}}{s^2+\omega_n^2} + \frac{\frac{t_0}{\omega_n^2}}{s} - \frac{\frac{t_0}{\omega_n^2}}{s^2+\omega_n^2}\right]$$

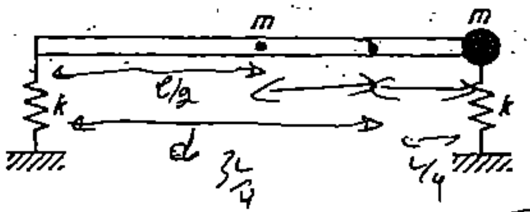
$$x(t) = \frac{f_0}{mt_0}\left[-\frac{1}{\omega_n^2}t + \frac{1}{\omega_n^3}\sin\omega_n t + \frac{t_0}{\omega_n^2} - \frac{t_0}{\omega_n^3}\sin\omega_n t\right]$$

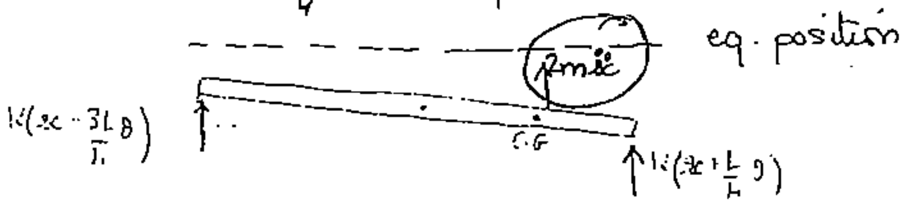
$$x(t) = \frac{f_0}{mt_0}\left[\frac{1-t_0}{\omega_n^3}\sin\omega_n t + \frac{t_0 - t}{\omega_n^2}\right].$$

X

2) (20 pts) A mass m is attached to the end of a uniform bar of length L and mass m also. If the system is supported by two springs, each having a stiffness k, as shown, determine the natural frequencies for small oscillations.

$$mg\frac{L}{2} + mgL = 2mgd$$
$$\Rightarrow d = \frac{3L}{4}$$





$$\sum F = -2m\ddot{x} - K(x + L/4\,\theta) \mp K\left(x - \frac{3L}{4}\theta\right)$$
$$\sum M_{C.G} = -K\left(x+\frac{L}{4}\theta\right)\left(\frac{L}{4}\right) + K\left(x - \frac{3L}{4}\theta\right)\left(\frac{3L}{4}\right) - \left(J_0 - m\frac{L^2}{16}\right)\ddot{\theta}$$
$$\Rightarrow \sum F = -2m\ddot{x} - Kx - K\frac{L}{4}\theta \mp Kx \oplus K\frac{3L}{4}\theta = 0$$
$$\Rightarrow m\ddot{x} + KL\theta = 0 \quad (1) \qquad 2m\ddot{x} + 2Kx - \frac{KL}{2}\theta = 0$$
$$\sum M_{C.G} = -\frac{KL}{4}x - \frac{KL^2}{16}\theta + \frac{3KL}{4}x - \frac{9KL^2}{16}\theta - \left(J_0 - m\frac{L^2}{16}\right)\ddot{\theta}$$
$$\Rightarrow \left(J_0 - m\frac{L^2}{16}\right)\ddot{\theta} + \frac{5KL^2}{8}\theta - \frac{KL}{2}x = 0 \quad (2) \qquad J_0 - m\frac{L^2}{16} = m\frac{L^2}{3} - m\frac{L^2}{16} = \frac{5mL^2}{16}$$

(1) and (2) constitute the eq. of motion, in matrix form:

$$\begin{pmatrix} 2m & 0 \\ 0 & \left(J_0 - \frac{mL^2}{16}\right) \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} 2K & -\frac{KL}{2} \\ -\frac{KL}{2} & \frac{5KL^2}{8} \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det(-M\omega^2 + K) = 0 \Rightarrow \det\begin{pmatrix} -m\omega^2 & KL \\ -KL/2 & \left(\frac{mL^2}{16} - J_0\right)\omega^2 + 5KL^2/8 \end{pmatrix} = 0$$

$$\Rightarrow \left(\frac{mL^2}{16} - J_0\right)\omega^2(-m\omega^2) + (-m\omega^2)\left(\frac{5KL^2}{8}\right) - \frac{K^2L^2}{2} = 0$$

$$\det(-M\omega^2 + K) = 0 \Rightarrow \det\begin{pmatrix} -2m\omega^2 + 2K & -KL/2 \\ -KL/2 & -\frac{5mL^2}{16}\omega^2 + \frac{5KL^2}{8} \end{pmatrix}$$

$$\Leftrightarrow -2(m\omega^2-k)\left(\frac{-5mL^2\omega^2+10kL^2}{16}\right)-\frac{k^2L^2}{4}=0$$

$$\Leftrightarrow 10(m\omega^2-k)\left(\frac{mL^2\omega^2-2kL^2}{16}\right)-\frac{k^2L^2}{4}=0$$

$$\Leftrightarrow \frac{5}{8}\left(m^2L^2\omega^4-2mkL^2\omega^2-mkL^2\omega^2-2k^2L^2\right)-\frac{k^2L^2}{4}=0$$

$$\Leftrightarrow \frac{5}{8}m^2L^2\omega^4-\frac{15}{8}mkL^2\omega^2-\frac{10}{8}k^2L^2-\frac{k^2L^2}{4}=0$$

$$\Leftrightarrow \frac{5}{8}m^2L^2\omega^4-\frac{15}{8}mkL^2\omega^2-\frac{3}{4}k^2L^2=0$$

Let $d=\omega^2$

$$\Rightarrow \frac{5}{8}m^2L^2d^2-\frac{15}{8}mkL^2d-\frac{3}{4}k^2L^2=0$$

$$\Delta=b^2-4ac=\frac{225}{64}m^2k^2L^4-\frac{15}{8}m^2k^2L^4=\frac{105}{64}m^2k^2L^4$$

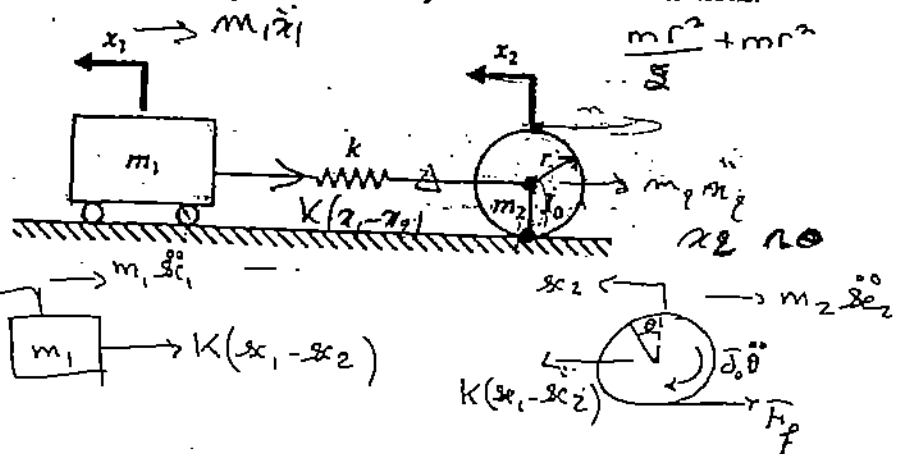
$$d_1=\frac{+\frac{15}{8}mkL^2+mkL^2\sqrt{\frac{105}{64}}}{\frac{5}{4}m^2L^2}=\frac{25.25}{8}mkL^2\times\frac{4}{5m^2L^2}$$

$$\Rightarrow d_1=2.525\frac{k}{m}\Rightarrow \omega_1=\sqrt{2.525\frac{k}{m}}=1.6\sqrt{\frac{k}{m}}$$

$$d_2=\frac{\frac{15}{8}mkL^2-mkL^2\sqrt{\frac{105}{64}}}{\frac{5}{4}m^2L^2}=\frac{4.75}{8}mkL^2\times\frac{4}{5m^2L^2}=0.475\frac{k}{m}$$

$$\Rightarrow \omega_2=0.67\sqrt{\frac{k}{m}}$$

3) (15 pts) Determine the natural frequencies of the system for small oscillations.



for m_1

$\sum F_x = -m_1\ddot{x}_1 - K(x_1 - x_2) \Rightarrow m_1\ddot{x}_1 + Kx_1 - Kx_2 = 0$ ①

$\sum M = 0$

for m_2 $\qquad m_2\ddot{x}_2 - K(x_1 - x_2) = 0$

$\sum F_x = -m_2\ddot{x}_2 \oplus K(x_1 - x_2) - F_f$

$\sum M_{center} = -m_2\ddot{x}_2 r - \bar{J}_0$

$-\bar{J}_0\ddot{\theta} + F_f r$

$x_2 = r\theta \Rightarrow F_f = \frac{\bar{J}_0\ddot{x}_2}{r^2}$

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$\Rightarrow \left(m_2 + \frac{\bar{J}_0}{r^2}\right)\ddot{x}_2 - Kx_1 + Kx_2 = 0$ ②

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 + \bar{J}_0/r^2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det(-M\omega^2 + K) = 0 \Rightarrow \det\begin{pmatrix} -m_1\omega^2 + K & -K \\ -K & -(m_2 + \bar{J}_0/r^2)\omega^2 + K \end{pmatrix} = 0$$

$$\Rightarrow (-m_1\omega^2 + K)\left[-\left(m_2 + \frac{\bar{J}_0}{r^2}\right)\omega^2 + K\right] - K^2 = 0$$

$$(-m_1\omega^2 + K)\left(-m_2\omega^2 - \frac{\bar{J}_0}{r^2}\omega^2 + K\right) - K^2 = 0$$

$$m_1 m_2\omega^4 + m_1\frac{\bar{J}_0}{r^2}\omega^4 - Km_1\omega^2 - Km_2\omega^2 - \frac{K\bar{J}_0}{r^2}\omega^2 = 0$$

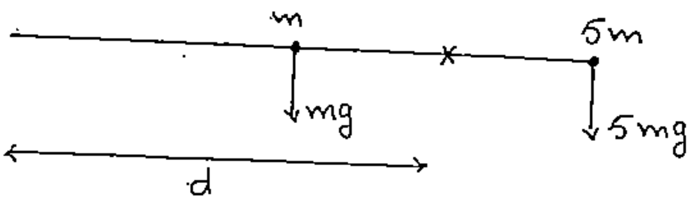
$(m_1 m_2 + m_1\bar{J}_0 \ldots)\omega^4 \ldots$

$$\Rightarrow \omega^2\left[\left(m_1 m_2 + m_1 \bar{J}_0/r^2\right)\omega^2 + K\left(-m_1 - m_2 - \frac{\bar{J}_0}{r^2}\right)\right] = 0$$

$$\Rightarrow \omega_1 = 0$$

$$\omega_2 = \frac{K\left(m_1 + m_2 + \bar{J}_0/r^2\right)}{\left(m_1 m_2 + m_1 \bar{J}_0/r^2\right)} = K\frac{\left(m_1 r^2 + m_2 r^2 + \bar{J}_0\right)}{\left(m_1 m_2 r^2 + m_1 \bar{J}_0\right)}$$

$$\Rightarrow \omega_2 = K\frac{\left(m_1 r^2 + m_2 r^2 + \bar{J}_0\right)}{m_1\left(m_2 r^2 + \bar{J}_0\right)}$$



$$\Rightarrow CG = mg\frac{L}{2} + 5mgL = 6mgd$$

$$\Rightarrow d_{C.G.} = \frac{11L}{12}$$