Summary - MATHS 201 – PART 1 (CHAPTERS 11 & 14)

I'll soon add chapters 15 and 16

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FINAL GRADE : 91

Note: the following summary is not infallible, it can help you to review and target the important points; but to get a good grade you have to solve a certain number of problems from the book !!

Done by Erik VZ (you can add me on facebook, if you have any question to ask me do not hesitate)

Please check out the following website!! http://4greeneraub.blogspot.com/ (about environmental issues)

If you are interested, join its page on Facebook, named For a Greener AUB

MATH 201

Chapter 11: Infinite sequences and series

1) Geometric series

$$\lim_{\infty} \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r|<1$$

$$\lim_{\infty} \sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1$$

2) The nth-Term test for divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{\infty} a_n$ fails to exist or is different from zero

If $\lim_{\infty} a_n = 0$ we don't know whether the series converges or diverges

3) Nondecreasing series / The integral test

→ A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above

* → Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge

$$\int_p^{n+1} f(x)\,dx \leq a_p + a_{p+1} + \dots + a_n \leq a_p + \int_p^n f(x)\,dx$$

* → The p-series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ It converges if $p>1$, It diverges if $p\leq 1$

4) The comparison test

Let $\sum a_n$ be a series with no negative terms

→ $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N

→ $\sum a_n$ diverges if there is a divergent series of nonnegat[ive] terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer [N]

5) The limit Comparison test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an inte[ger])

→ If $\lim\limits_{\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both conve[rge] or both diverge

→ If $\lim\limits_{\infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ conver[ges]

→ If $\lim\limits_{\infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverge[s]

6) The ratio test

Let $\sum a_n$ be a series with positive terms and suppose tha[t]

$\lim\limits_{\infty} \frac{a_{n+1}}{a_n} = \ell$

Then.

(a) the series converges if $\ell < 1$

(b) the series diverges if $\ell > 1$ or $\ell \to \infty$

(c) the test is inconclusive if $\ell = 1$

7) The root test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ and suppose that $\lim\limits_{\infty} \sqrt[n]{a_n} = \ell$

Then,

(a) the series converges if $\ell < 1$

(b) the series diverges if $\ell > 1$ or $\ell \to \infty$

(c) the test is inconclusive if $\ell = 1$

8) The Alternating Series Test (Leibniz's Theorem)

The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$ converge[s] if all three of the following conditions are satisfied:

1. The u_n's are all positive

2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N (decreasin[g])

3. $u_n \to 0$

9) The Alternating series estimation theorem

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the three conditions (view before) and is converging, then for $n \geq N$, $S_n = u_1 - u_2 + \dots + (-1)^{n+1} u_n$ approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value for the first unused term. Furthermore, the remainder, $L - S_n$, has the same sign as the first unused term.

10) The Absolute convergence test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

(A series that converges but does not converge absolutely converges conditionally).

11) The rearrangement theorem for Absolutely convergent series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \dots b_n, \dots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

12) The convergence theorem for power series

* a power series about $x = a$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots c_n(x-a)^n + \dots$$

x and a are constants, a is called the center

(use ratio test) * If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ converges for $x = c \neq 0$, then it converges absolutely for all $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$

13) The radius of convergence of a power series

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three possibilities

1. there is a positive number R such that the series diverge[s] for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$. The series may or may no[t] converge at either of the endpoints $x=a-R$ an[d] $x=a+R$
2. the séries converges absolutely for every x $(R=\infty)$
3. the series converges at $x=a$ and diverges elsewhe[re]

$\rightarrow$ R is called the radius of convergence of the power series and the interval of radius R centered at $x=a$ is called the interval of convergence.

14) The term by term differentiation theorem

If $\sum c_n (x-a)^n$ converges for $a-R<x<a+R$ for some $R>0$, it defines a function f:

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n, \quad a-R<x<a+R = I$$

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

$(\ldots)$

Each of these derived series converge for $x \in I$

15) The term by term Integration theorem

Suppose that $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$

converges for $a-R<x<a+R$ $(R>0)$. Then,

$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $a-R<x<a+R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

for $a-R<x<a+R$

16) Definition - Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

The Maclaurin series generated by f is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

the Taylor series generated by f at $x = 0$

17) Definition – Taylor polynomial of order n

Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

18) Taylor's theorem (generalization of the mean value theorem)

If f and its first n derivatives $f', f'', \dots f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

by f at $x=a$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

the Taylor series generated by f at $x=0$

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Let f be a function with derivatives of order k for $k=1,2,\ldots,N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x=a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

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If f and its first n derivatives $f', f'', \ldots f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

9 Taylor's formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a a

(for each $x \in I$, $f(x) = P_n(x) + R_n(x)$

$R_n(x)$ is the remainder of order n

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$,
we say that the Taylor series generated by f at $x=a$ converges to f on I

We write:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Often we can estimate R_n without knowing the value o

10 The remainder estimation theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

(for $n \to \infty$ $\quad M_x \dfrac{|x-a|^{n+1}}{(n+1)!} \to 0$

$R_n(x) \to 0$)

Functions to learn

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots \text{ for all } x$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \text{ for all } x$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \text{ for all } x$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^n \quad \text{for } x \in]-1;1[$$

21] The binomial series

For $-1 < x < 1$

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

where we define $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$,

and $\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$ for $k \geq 3$

$f(x) = (1+x)^m$

$$T.S = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots + \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$$

$T.S. = f(x)$ for $-1 < x < 1$

If $m = -1$

$$\binom{-1}{1} = -1, \quad \binom{-1}{2} = \frac{(-1)(-2)}{2!} = 1,$$

and $$\binom{-1}{k} = \frac{-1(-2)(-3)\dots(-1-k+1)}{k!}$$

$$= (-1)^k \left(\frac{k!}{k!}\right) = (-1)^k$$

$$(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k x^k = 1 - x + x^2 - \dots + (-1)^k x^k + \dots$$

22) Fourier Series

Suppose that we want to approximate a function $f(x)$ in $[0; 2\pi]$ by a sum of sines and cosines

$$f_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx \quad k = 1, 2, \dots, n$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx \quad k = 1, 2, \dots, n$$

If $n \to \infty$, then $f_n(x) = f(x)$ and the resulting sum is called Fourier series for $f(x)$

Formulas:

$$\int_0^{2\pi} \cos px\, dx = 0 \text{ for all } p$$

$$\int_0^{2\pi} \sin px\, dx = 0 \text{ for all } p$$

$$\int_0^{2\pi} \cos px \cos qx\, dx = \begin{cases} 0 & \text{if } p \neq q \\ \pi & \text{if } p = q \end{cases}$$

$$\int_0^{2\pi} \sin px \sin qx\, dx = \begin{cases} 0 & \text{if } p \neq q \\ \pi & \text{if } p = q \end{cases}$$

$$\int_0^{2\pi} \sin px \cos qx\, dx = 0 \text{ for all } p \text{ and } q$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1 \rightarrow 2\cos^2 x = \cos 2x + 1$$

$$\boxed{\cos^2 x = \frac{\cos 2x + 1}{2}}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$$

A function $f(x)$ is piecewise continuous on an interval I if $f(x)$ has a finite number of discontinuities on I

23) Convergence of Fourier series

Let $f(x)$ be a function such that f and f' are piecewise continuous on the interval $[0, 2\pi]$. Then f is equal to its Fourier series at all points where f is continuous. At a point c where f has a discontinuity, the Fourier series converges to:

$$\frac{f(c^+) + f(c^-)}{2}$$ where $f(c^+)$ and $f(c^-)$ are the right-and left-hand limits of f at c

Chapter 10.5 : Polar Coordinates

While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates

$P(r, \theta) \rightarrow$ polar coordinates

- Equation :

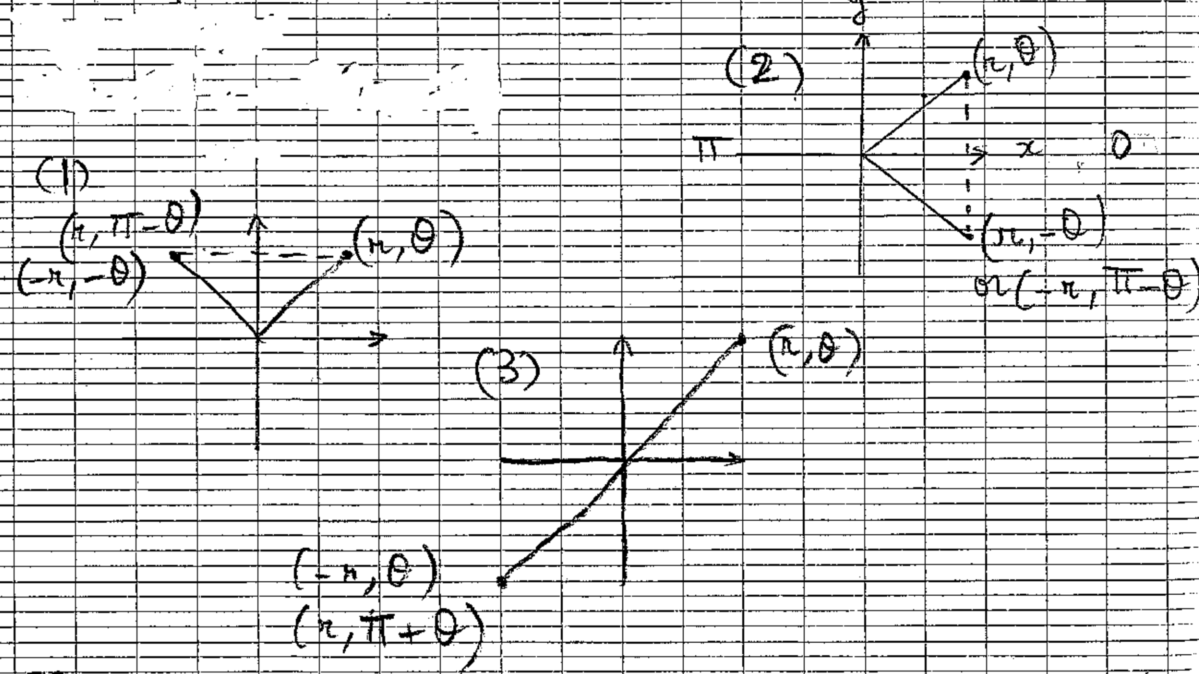
$r = a \iff$ circle radius $|a|$ centered at O

$\theta = \theta_0 \iff$ line through O making an angle θ_0 w the initial ray

- Equations relating polar and cartesian coordinates

$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad x^2 + y^2 = r^2$

Graphing in polar coordinates



- Slope :

The slope of a polar curve $r = f(\theta)$ is given by $\frac{dy}{dx}$ not by $r' = \frac{df}{d\theta}$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$$\text{Slope} = \frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{d/d\theta\,(f(\theta)\cos\theta)}$$

Symmetry tests for polar graphs :

1) Symmetry about the x-axis :
If (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi-\theta)$ lies on the graph

2) Symmetry about the y-axis $(r, \pi-\theta), (-r, -\theta)$

3) Symmetry about the origin $(-r, \theta), (r, \theta+\pi)$

Chapter 13 : Vector _ Valued Functions and Motion in spac

Vector Functions

$P(x, y, z)$

$\vec{r}(t) = \overrightarrow{OP} = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$

At time t, it's the position vector (it's a curve in spa that we call particle path)

Def: $r(t) = \langle x(t), y(t), z(t) \rangle$

$\lim\limits_{t \to t_0} r(t) = \langle \lim\limits_{t \to t_0} x(t), \lim\limits_{t \to t_0} y(t), \lim\limits_{t \to t_0} z(t) \rangle = \vec{L}$

Continuity: $r(t) = \langle x(t), y(t), z(t) \rangle$ is continuous at a poi $t = t_0$ if $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$

Each component must be continuous.

Differentiability: $\vec{r}'(t) = \lim\limits_{\Delta t \to 0} \dfrac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t}$

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

A vector function $\vec{r}$ is differentiable if it is differenti at every point of its domain

The curve traced by r is smooth if $\frac{d\vec{r}}{dt}$ is continuo and never 0

Velocity

1) velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = r'(t)$

2) speed is the magnitude of the velocity $= |\vec{v}|$

$|\vec{v}| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2} =$ speed

3) acceleration $= \dfrac{d^2\vec{r}}{dt^2} = \vec{r}''(t) = v'(t)$

$\to \vec{r} \cdot \dfrac{d\vec{r}}{dt} = 0 \Leftarrow$ $\vec{r}$ is a differentiable vector function of t of constant length

$$\int f(u)\, u'\, dt = \int f(u)\, du$$

Example: when we track a particle moving on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere. Then $\frac{d\vec{r}}{dt}$ (velocity) is tangent to the sphere and perpendicular to $\vec{r}$ (dot product)

Differentiation rules for vector functions: (same rules as seen before) ⊕

Dot product rule

$$\frac{d}{dt}[u(t)\cdot v(t)] = u'(t)\cdot v(t) + u(t)\cdot v'(t)$$

⚠ Remember to preserve the order of the factors

Cross product rule

$$\frac{d}{dt}[u(t)\times v(t)] = u'(t)\times v(t) + u(t)\times v'(t)$$

Chain rule

$$\frac{d}{dt}[u(f(t))] = f'(t)\, u'(f(t))$$

Definite Integral

If the component of $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ are integrable over $[a;b]$ then so is $\vec{r}$:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\vec{i} + \left(\int_a^b y(t)\,dt\right)\vec{j} + \left(\int_a^b z(t)\,dt\right)\vec{k}$$

Length of a smooth curve

The length of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$ is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$L = \int_a^b |v|\,dt$$

Chapter 14 : Partial derivatives

14.1) • Function of n independent variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2,$ ... $x_n)$. A real-valued function f on D is a rule that assigns a unique single real number
$w = f(x_1, x_2, \ldots x_n)$ to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables x_1 to x_n. We also call to x_j's the function's input variables and call w the function's output variable.

• Functions of two variables

Interior and boundary points, open, closed

- A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if it is the center of a disk of positive radius that lies entirely in R.
- A point (x_0, y_0) is a boundary point of R if every d[isk] centered at (x_0, y_0) contains points that lie outside o[f R]
- The interior points of a region, as a set, make up the in[terior] of the region. The region's boundary points make up its boundary. A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary points.
- A region in the plane is bounded if it lies inside a disk of fixed radius. A region is unbounded if it is [not] bounded.

Domain / Range

domain : $x < y$ (example)

range : $[-1 ; 2]$

Level Curve, Graph, Surface

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=C$ is called a level curve of f. The set of all points $(x, y, f(x,y))$ in space, for (x,y) in the domain of f, is called the graph of f. The graph of f is also called the surface $z = f(x,y)$

14.2 Limits and continuity in higher dimension

Definition : Limit of a function of two variables

We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Theorem 1 : Properties of limits of functions of two variables

L, M and k are real number

(as in the plane)

Definition : Continuous function of two variables

A function $f(x,y)$ is continuous at the point (x_0, y_0) if :

1) f is defined at (x_0, y_0)
2) $\lim_{(x_0,y_0)} f(x,y)$ exists
3) $\lim_{(x_0,y_0)} f(x,y) = f(x_0, y_0)$

We have to find that the limit doesn't exist (more easy)

A function is continuous if it is continuous at every point of its domain.

Two Path Test for nonexistence of a limit

If a function $f(x,y)$ has different limits along two different paths as (x,y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Continuity of composites

If f is continuous at (x_0, y_0) and g is a single variable function continuous at $f(x_0,y_0)$, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f($ is continuous at (x_0, y_0)

14.3 Partial Derivatives

Def: Partial derivative with respect to x:

The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exist.

Def: Partial derivative with respect to y

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h} = f_y$$

means: intersection of the plane $x = x_0$ (constant) with the surface $z = f(x,y)$ which gives us a plane with a curve $z = f(x_0, y)$. The partial derivative with respect to y is the derivative of this function (the slope of this curve).

Notations: $\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0,y_0) = \frac{\partial f}{\partial x} = f_x$

→ A function can be discontinuous and its partial derivative may exist!

The mixed derivative theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then: $f_{xy}(a,b) = f_{yx}(a,b)$

The increment theorem for functions of two variables

Suppose that the first partial derivatives of $f(x,y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of ε_1 and $\varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

(this theorem holds also for severable variables)

Differentiable function (definition)

A function $z = f(x,y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

We call f differentiable if it is differentiable at every point in its domain

Continuity of partial derivatives implies differentiability

If f_x and f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Differentiability implies continuity

If a function $f(x,y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

14.4 The Chain Rule

Chain Rule for functions of two independent variables

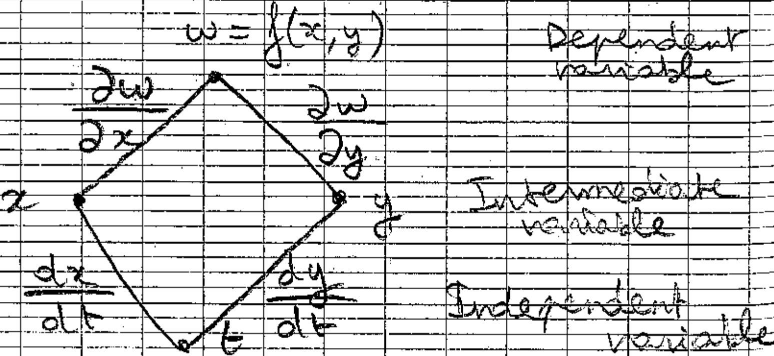
If $\omega = f(x,y)$ has continuous partial derivatives f_x and and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $\omega = f(x(t), y(t))$ is a differentiable function of dt and:

$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

or, $$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\left.\frac{dw}{dt}\right|_{t_0} = \left.\frac{\partial w}{\partial x}\right|_{P_0} \times \left.\frac{dx}{dt}\right|_{t_0} + \left.\frac{\partial w}{\partial y}\right|_{P_0} \left.\frac{dy}{dt}\right|_{t_0} \qquad P_0 = (x_0,$$

Remember: Chain Rule



Chain Rule for functions of three Independent variables

If $w = f(x, y, z)$ is differentiable and $x, y,$ and z are differentiable functions of t, then w is a differentiable function of t and:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

3 intermediate var

Chain rule for two independent variables and three Intermediate variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$, if all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

And, $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Dependent variable — w

Intermediate variable — $x \quad y \quad z$

Independent variable — r, s

$$w = f\big(g(r,s),\ k(r,s),\ h(r,s)\big)$$

2 intermediate var

$\rightarrow$ If $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

And, $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

1 intermediate var

If $w = f(x)$ and $x = g(r, s)$, then,

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Implicit differentiation revisited

We suppose that 1) The function $F(x,y) = w$ is differentiable

2) The equation $F(x,y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$

$$0 = \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$= F_x \times 1 + F_y \times \frac{dy}{dx}$$

If $F_y = \frac{\partial w}{\partial y} \neq 0 \Leftrightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$

(Shortcut to finding derivatives of implicitly defined functions which we state here as a theorem.

General case

Functions of many variables + Chain rule

(Think of the dot product of two vectors)

$\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots, \frac{\partial w}{\partial v}\right)$ and $\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \dots, \frac{\partial v}{\partial p}\right)$

derivatives of w with respect to the intermediate variables — derivatives of the intermediate variables with respect to selected independent variables

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \dots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$

[...]

14.5) Directional Derivative and Gradient Vectors

Definition: Directional Derivative

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the number

$$(D_u f)_{P_0} = \left.\frac{df}{ds}\right|_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

"The derivative of f at P_0 in the direction of $\vec{u}$"

The directional derivative is only a generalization of the partial derivative

$D_u f$ with $u = \langle 1, 0\rangle \Leftrightarrow \frac{\partial f}{\partial x}$ | $u = \langle 0, 1\rangle$ $D_u f = \frac{\partial f}{\partial y}$

It represents the (instantaneous) rate of change of f at P_0 in the direction of $\vec{u}$.

Definition: Gradient Vector

The gradient vector (gradient) of $f(x,y)$ at a point $P_0(x_0, y_0)$ is the vector

$\vec{\nabla f} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ obtained by evaluating the partial derivatives of f at P_0.

Thm: The Directional Derivative is a dot product

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$D_u f\Big|_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}$ dot product of the gradient ∇f at P_0 and $\vec{u}$

Properties of the directional derivative $D_u f = \nabla f \cdot u$
$|u| = 1$ $= |\nabla f| \cos\theta$

$D_u f = \nabla f \cdot \vec{u} = |\nabla f| \cos\theta$

1) The function f increases most rapidly when $\cos\theta = 1$ or when $\vec{u}$ is the direction of ∇f. That is, at each P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is:

$D_u f = |\nabla f| \cos(0) = |\nabla f|$

2) Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is:

$D_u f = |\nabla f| \cos(\pi) = -|\nabla f|$

3) Any direction $\vec{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and

$D_u f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = |\nabla f| \cdot 0 = 0$

These properties hold in three dimensions as well as two.

Gradients and Tangents to level curves

At every point (x_0, y_0) in the domain of a differentiable function $f(x,y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Proof: if $\vec{r} = x(t)\vec{i} + y(t)\vec{j}$ (differentiable function)
and $f(x(t), y(t)) = c$

Then, we differentiate this equation:

$$\Leftrightarrow \frac{d}{dt}\left(f(x(t), y(t))\right) = \frac{d}{dt} c$$

$$\Leftrightarrow \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\Leftrightarrow \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right) \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right) = 0$$

$$\Leftrightarrow \vec{\nabla f} \cdot \underbrace{\frac{d\vec{r}}{dt}}_{\text{velocity}} = 0$$

Therefore, $\vec{\nabla f}$ is normal to $\frac{d\vec{r}}{dt}$ which is the tangent to the curve, the $\vec{\nabla f}$ is normal to the curve.

This allow us to find equations for tangent lines to level curves (lines normal to gradients)

$\vec{N} = A\vec{i} + B\vec{j} \Rightarrow D: A(x - x_0) + B(y - y_0) = 0$

Then, If $\vec{N} = \vec{\nabla f}_{(x_0, y_0)} = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$

the equation is the tangent line given by:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Algebra Rules for Gradients

1) Constant multiple rule $\nabla(kf) = k\nabla f$
2) Sum rule $\nabla(f + g) = \nabla f + \nabla g$
3) Difference rule $\nabla(f - g) = \nabla f - \nabla g$
4) Product Rule $\nabla(fg) = f\nabla g + g\nabla f$
5) Quotient Rule $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Function of three variables

$f(x,y,z)$ differentiable

$$\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

and

$$\frac{df}{ds} = D_u f = \vec{\nabla f}\cdot\vec{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3$$

$$D_u f = \vec{\nabla f}\cdot\vec{u} = |\nabla f|\,|u|\cos\theta = |\nabla f|\cos\theta$$

$\boxed{14.6}$ Tangent Planes and differentials

Definition: Tangent Plane, Normal Line

The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Equations: level surface

Tangent Plane to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$

$$f_x(P_0)\times(x-x_0) + f_y(P_0)\times(y-y_0) + f_z(P_0)\times(z-z_0) = 0$$

Normal Line to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$

$$\begin{cases} x = x_0 + f_x(P_0)\times t \\ y = y_0 + f_y(P_0)\times t \\ z = z_0 + f_z(P_0)\times t \end{cases}$$

Line // ∇f

Plane tangent to a surface $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$

The plane tangent to the surface $z = f(x,y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0)$

$P = (x_0, y_0, f(x_0,y_0))$ is:

$$f_x(x_0,y_0)\times(x-x_0) + f_y(x_0,y_0)\times(y-y_0) - (z-z_0) = 0$$

Estimating the Change in f in a Direction $\vec{u}$

To estimate the change in the value of a differenti[able] function f when we move a small distance ds from a point P_0 in a particular direction $\vec{u}$, we use the formula

$$df = \underbrace{\left(\vec{\nabla} f|_{P_0} \cdot \vec{u}\right)}_{\text{Directional derivative}} \cdot \underbrace{ds}_{\text{distance increment}}$$

How to linearize a function of two variables

Definition : Linearization, Standard Linear Approximation

The linearization of a function $f(x,y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x,y) = f(x_0, y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The approximation $f(x,y) \simeq L(x,y)$ is the standard linear approximation of f at (x_0, y_0).

The linearization of a function of 2 variables is a tangent plane approximation.

The error in the standard linear approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x,y)$ incurred in replacing $f(x,y)$ on R by its linearization

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

satisfies the inequality

$$|E(x,y)| \leq \frac{1}{2} M \left(|x-x_0| + |y-y_0|\right)^2$$

Definition: Total differential (estimations)

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change $df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$ in the linearization of f is called the total differential of f.

Functions of more than two variables (analogous results hold)

1) The linearization of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is:

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$$

2) Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$ and $|f_{yz}|$ are all less than or equal to M throughout R. Then the error $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| < \frac{1}{2} M\left(|x - x_0| + |y - y_0| + |z - z_0|\right)^2$$

3) If the second partial derivatives of f are continuous and if x, y and z change from x_0, y_0 and z_0 by small amounts dx, dy and dz, the total differential

$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

gives a good approximation of the resulting change in f.

14.7 Extreme values and saddle points

Def: Local Maximum, Local minimum

Let $f(x,y)$ be defined on a region R containing the po
(a,b). Then

1) $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,$
for all domain points (x,y) in an open disk centered
(a,b)

2) $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,$
for all domain points (x,y) in an open centered disk

At such points, the tangent planes, when they exist are horizontal

Local extrema are also called relative extrema.

Thm: First Derivative Test for local extreme values

If $f(x,y)$ has a local maximum or minimum value
an interior point (a,b) of its domain and if the first partial derivatives exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Equation of tangent plane: $z = f(a,b)$ (horizontal)

Def: Critical Point

An interior point of the domain of a function $f(x,y)$ whe
both f_x and f_y are zero or where one or both of f_x
f_y do not exist is a critical point of f.

Def: Saddle Point

A differentiable function $f(x,y)$ has a saddle point a
a critical point (a,b) if in every open disk centered
(a,b) there are domain points (x,y) where $f(x,y) > f($
and domain points (x,y) where $f(x,y) < f(a,b)$. The corres-
ding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ i
called a saddle point of the surface.

Thm: Second derivative test for local extreme values

Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then,

discriminant

1) $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b)

a) $f_{xx} < 0 \rightarrow$ local maximum Concave

b) $f_{xx} > 0 \rightarrow$ local minimum Convex

2) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,b) \rightarrow$ saddle point

3) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,b) \rightarrow$ the test is inconclusive

Absolute maxima and minima on closed bounded regions

We organize the search for the absolute extrema of a continuous function $f(x,y)$ on a closed and bounded region R into three steps:

1) List of the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f

2) List the boundary points of R where f has local maxima and minima and evaluate f at these points.

3) Find absolute minima and maxima.

(not easy at all, we have to analyse every point)

14.8) Lagrange Multipliers

Thm: The Orthogonal Gradient Theorem

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve

$C: r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then $\vec{\nabla f}$ is orthogonal to C at P_0.

⇒ At the points on a smooth curve $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j}$ w
a differentiable function $f(x,y)$ takes on its local ma
and minima relative to its values on the curve, $\vec{\nabla} f \cdot \vec{v}$
where $\vec{v} = \dfrac{d\vec{r}}{dt}$

The method of Lagrange Multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiabl
and $\vec{\nabla} g \neq \vec{0}$ when $g(x,y,z) = 0$. To find the local maximum and minimum values of f subjected to the constraint $g(x,y,z) = 0$ (if these exist), find the values of x, y, z and λ that simultaneously satisfy the equation

$$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \text{and} \quad g(x,y,z) = 0$$

For functions of two independent variables, the cond
is similar, but without the variable z.

By Erik VZ

tell me if it was useful for you !

Don't forget to check out http://4greeneraub.blogspot.com/ (about environmental issues)

If you are interested, join its page on Facebook, named "For a Greener AUB"