American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2008-2009

quiz # 2 solution

1. Find the domain and the range of the function $f(x,y) = \sqrt{xy}$. Determine the boundary of the domain, is the domain open or closed, also determine if the domain is bounded or unbounded. Find the equation of the level curve passing through $(1,1)$. Determine all the level curves of f.

$D_f = \{(x,y);\ x \geq 0 \textit{ and } y \geq 0 \textit{ or } x \leq 0 \textit{ and } y \leq 0\}$; Range=$[0,\infty[$; boundary: $\{x=0\}$ and $\{y=0\}$; domain is closed (it contains its boundary); and unbounded.

Level curve passing through $(1,1)$: $f(1,1)=1$ hence the level curve passing through $(1,1)$ is defined by $\sqrt{xy}=1$, which gives $y=1/x$

Level curves: $f(x,y)=c$ where c is in the range of f, hence $\sqrt{xy}=c$, and then $y=c^2/x$ are all the level curves of f

2. a. Give the first four terms of the Maclaurin series of $f(x)=\sqrt{1-x^2}$, then deduce $f^{(4)}(0)$.
$(1-x^2)^{1/2} = 1 - x^2/2 - x^4/8 - x^6/16 + ..$ (by using the binomial series), and hence $f^{(4)}(0) = -4!/8$

b.
$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sin x}{x^3} - \frac{1}{x^2} + a\right) &= \lim_{x\to 0}\left(\frac{x - x^3/6 + o(x^5)}{x^3} - \frac{1}{x^2} + a\right)\\ &= \lim_{x\to 0}\left(\frac{1}{x^2} - 1/6 + o(x^2) - \frac{1}{x^2} + a\right)\\ &= \lim_{x\to 0}\left((a-1/6) + o(x^2)\right) = a - 1/6\end{aligned}$$

hence the limit is equal to 0 if and only if $a = 1/6$.

3. a. Let $w = x^2e^{2y}\cos(3z)$. Use the chain rule to find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x = \cos t, y = \ln(t+2), z = t$.

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt}\\ &= 2xe^{2y}\cos(3z)(-\sin t) + 2x^2e^{2y}\cos(3z).\frac{1}{t+2} - 3x^2e^{2y}\sin(3z)\end{aligned}$$

at $(1,\ln 2, 0)$, $t=0$, and hence, $\dfrac{dw}{dt}(1,\ln 2,0) = 4$.

b. Find the equation of the tangent plane to the surface $z = y - x^2$ at $(1,2,1)$.
Let $w = y - x^2 - z$, $\nabla w = (-2x, 1, -1)$, then $\nabla w(1,2,1) = (-2,1,-1)$, and $(T): -2(x-1) + (y-2) - (z-1) = 0$ is the tangent plane.

4. Find the absolute minimum and maximum of the function $f(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x=0, y=4, y=x$.

critical point: $\partial f/\partial x = 2x - y$ and $\partial f/\partial y = -x + 2y$, then the only critical point is $(0,0)$

on the path x=0: $g(y) = f(0,y) = y^2 + 1$, $g'(y) = 2y$, $g'(y) = 0 \rightarrow y = 0$

on the path y=4: $h(x) = f(x,4) = x^2 - 4x + 17$, $h'(x) = 2x - 4$, $h'(x) = 0 \to x = 2$

on the path y=x: $k(x) = f(x,x) = x^2 + 1$, $k'(x) = 2x$, $k'(x) = 0 \to x = 0$

point	value
(0,0)	1
(2,4)	13
(0,4)	17
(4,4)	17

The minimum is then equals to 1 at $(0,0)$; the maximum is equals to 17 at $(0,4)$ and $(4,4)$

5. a. Find the direction of maximum increase of $f(x,y) = x^2 - 3xy + 4y^2$ at $P(1,2)$. Is there is a direction $\mathbf{u}$ in which the rate of change of f at $P(1,2)$ equals 14? Justify your answer.
$\nabla f = (2x - 3y, -3x + 8y)$, then $\nabla f(1,2) = (-4, 13)$, and $|\nabla f(1,2)| = \sqrt{16 + 169} = \sqrt{185}$
The direction of maximum increase of f is the direction of $\nabla f(1,2)$, hence $(-4/\sqrt{185}, 13/\sqrt{185})$

The rate of change in the direction of maximum increase is $\sqrt{185} < 14$, then there is no direction $\mathbf{u}$ in which the rate of change of f at $P(1,2)$ equals 14.

b. Find the points on the surface $(y+z)^2 + (z-x)^2 = 16$ where the normal line is parallel to the yz-plane.
Let $f(x,y,z) = (y+z)^2 + (z-x)^2$
$\nabla f = (-2(z-x), 2(y+z), 2(y+z) + 2(z-x))$
If the normal line is parallel to the yz-plane, then x is constant, and $\partial f/\partial x = 0$, hence $-2(z-x) = 0$, and then $x = z$; as the point $P(x,y,z)$ belongs to the surface, then $(y+z)^2 = 16$ or $y + z = \pm 4$
Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t \pm 4, t)$ where t is a real number.

6. a. Show that the limit of $f(x,y) = \dfrac{3x^2y}{x^2 + 4y^2}$ exists at $(0,0)$.
$|f(x,y)| \le |\frac{3x^2y}{x^2}| = |3y| \longrightarrow 0$ as $(x,y) \to (0,0)$, hence by sandwich theorem, $f(x,y) \longrightarrow 0$

b. Show that $f(x,y) = \dfrac{\sin(x-y)}{|x| + |y|}$ has no limit at $(0,0)$.
consider the path $y = 0$, note that this path passes through $(0,0)$
$f(x,0) = \dfrac{\sin(x)}{|x|} \longrightarrow \pm 1$ as $x \to 0$;
the limit does not exist, then by the two path test, $f(x,y)$ has no limit as $(0,0)$.