EEN 344
Communication Systems I
Class Notes

Chapter 2

Fourier Representation of Signals and Systems

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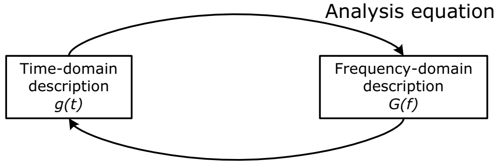
The Fourier Transform

- A signal is a function of time, but from a communications system perspective, it is important that we know also the frequency content of the signal.
- The Fourier transform relates the frequency-domain description of a signal to its time-domain description.
- Mainly two versions:
  - Continuous Fourier transform (FT), for continuous functions in both time and frequency domains
  - Discrete Fourier transform (DFT), for discrete data in both time and frequency domains.
- In this chapter, we will concentrate on FT to determine the frequency content of a continuous-time signal.
- Evaluates what happens to this frequency content when the signal is passed through a linear time-invariant LTI system.

Definition of the FT

Analysis equation: $$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$



Synthesis equation: $$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df$$

Note: $e^{-j2\pi f t} = \cos(2\pi f t) - j\sin(2\pi f t)$

- Advantages of using frequency-domain analysis
  - Provides the frequency content of a signal.
  - Resolution into eternal sinusoids presents the behavior as the superposition of steady-state effects.
  - If the time-domain analysis involves solving differential equations, the frequency domain involves simple algebraic equations.

Existence of FT

- For the FT of a signal *g(t)* to exist, it is sufficient, but not necessary, that the function *g(t)* (Dirichlet's Conditions):
  - is single-valued, with a finite number of maxima and minima in any finite time interval.
  - has a finite number of discontinuities in any finite time interval.
  - is absolutely integrable:

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$$

Existence of FT II

- If the signal is physically realizable, then the FT of this signal exists. In order for the signal to be physically realizable, its energy

$$\int_{-\infty}^{\infty} |g(t)|^2 dt$$

must satisfy

$$\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$$

- In this case, such a signal is called an energy signal. Therefore, all energy signals are Fourier transformable.

Continuous Spectrum

- The FT is a complex function of frequency so that

$$G(f) = |G(f)| e^{j\theta(f)}$$

where

$|G(f)|$ is the continuous amplitude spectrum

$\theta(f)$ is the continuous phase spectrum

- If *g(t)* is a real-valued function of time, the FT has the following characteristics

$G(-f) = G^*(f)$ (where the asterisk denotes complex conjugation) $(x + jy)^* = x - jy$

$|G(-f)| = |G(f)|$

$\theta(-f) = -\theta(f)$

Continuous Spectrum II

- In conclusion
  - The amplitude spectrum of a signal is an even function of the frequency; the amplitude spectrum is symmetric with respect to the origin *f*=0.
  - The phase spectrum of a signal is an odd function of the frequency; the phase spectrum is odd-symmetric with respect to the origin *f*=0

Properties of the FT

1. Linearity

$$c_1 g_1(t) + c_2 g_2(t) \Leftrightarrow c_1 G_1(f) + c_2 G_2(f)$$

2. Dilation

$$g(at) \Leftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

3. Conjugation

$$g^*(t) \Leftrightarrow G^*(-f)$$

4. Duality

If $g(t) \Leftrightarrow G(f)$, then $G(t) \Leftrightarrow g(-f)$

Properties of the FT II

5. Time Shifting

$$g(t-t_o) \Leftrightarrow G(f)e^{-j2\pi f t_o}$$

6. Frequency Shifting

$$e^{j2\pi f_c t} g(t) \Leftrightarrow G(f-f_c)$$

7. Area under *g(t)*

$$\int_{-\infty}^{\infty} g(t)dt = G(0)$$

8. Area under *G(f)*

$$g(0) = \int_{-\infty}^{\infty} G(f)df$$

Properties of the FT II

9. Differentiation in the time domain

$$\frac{d^n}{dt^n}\{g(t)\} \Leftrightarrow (j2\pi f)^n G(f)$$

10. Integration in the time domain

$$\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \frac{1}{j2\pi f}G(f)$$

11. Modulation Theorem

$$g_1(t)g_2(t) \Leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda$$

Properties of the FT III

12. Convolution Theorem

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \Leftrightarrow G_1(f)G_2(f)$$

Shorthand notation: $g_1(t) * g_2(t) \Leftrightarrow G_1(f)G_2(f)$

13. Correlation Theorem (assuming that $g_1(t)$ and $g_2(t)$ are complex valued)

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t-\tau)dt \Leftrightarrow G_1(f)G_2^*(f)$$

Properties of the FT IV

14. Rayleigh's Energy Theorem

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

The Inverse Relationship Between Time and Frequency

- If the time-domain description of a signal is changed, the frequency-domain description of the signal is changed in an inverse manner.
- If a signal is strictly limited in frequency, the time-domain description of the signal will trail on indefinitely, even though its amplitude may assume a progressively smaller value.
- In an inverse manner, if a signal is strictly limited in time, then the spectrum of the signal is infinite in extent, even though the amplitude spectrum may assume a progressively smaller value.
- Accordingly, a signal cannot be strictly limited in both time and frequency.

Dirac Delta Function

- The theory of the FT is applicable to only time functions that satisfy the Dirichlet conditions, but it would be helpful to extend the theory in two ways:
  - To combine the theory of Fourier series and FT, so that the Fourier series may be treated as a special case of the FT.
  - To expand applicability of the FT to include power signals (periodic signals), signals that satisfy:

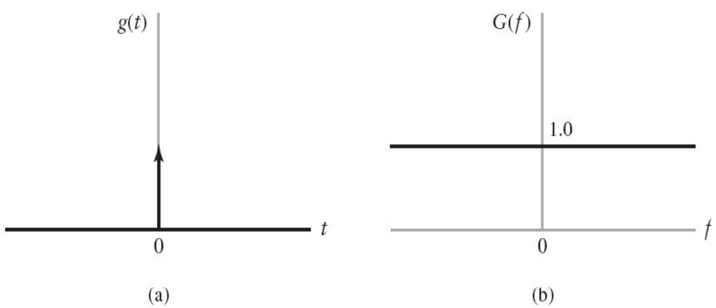
$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt\right\}<\infty$$

Dirac Delta Function II

- This can be accomplished by the use of the Dirac Delta function: $\delta(t)$

$$\delta(t)=0,\quad t\neq 0$$

$$\int_{-\infty}^{\infty}\delta(t)\,dt=1$$



(a) The Dirac delta function δ(t). (b) Spectrum of δ(t).

Dirac Delta Function III

$$\int_{-\infty}^{\infty}g(t)\delta(t-t_o)\,dt=g(t_o)$$ (sifting property)

Since $\delta(t)$ is an even function of t

$$\int_{-\infty}^{\infty}g(\tau)\delta(t-\tau)\,d\tau=g(t)$$

therefore $g(t)*\delta(t)=g(t)$

Fourier Transform: $\delta(t)\Leftrightarrow 1$

Applications of the Delta Function

- DC signal

$$1 \Leftrightarrow \delta(f)$$

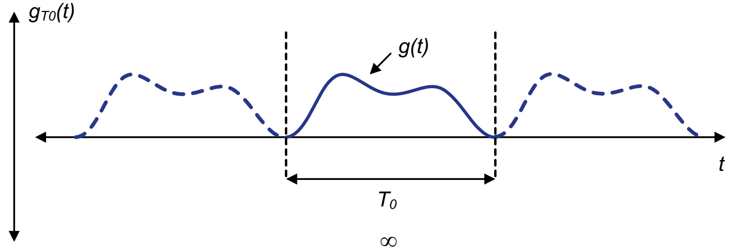
- Complex exponential

$$e^{j2\pi f_c t} \Leftrightarrow \delta(f - f_c)$$

- Sinusoidal functions

$$\cos(2\pi f_c t) \Leftrightarrow \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

FT of Periodic Signals



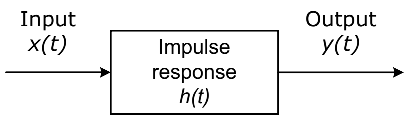
$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_o)$$

$$g_{T_0}(t) \Leftrightarrow f_o \sum_{n=-\infty}^{\infty} G(nf_o)\delta(f - nf_o)$$

where $G(nf_o) = \int_{-\infty}^{\infty} g(t) e^{(-j2\pi n f_0 t)} dt$

Transmission of Signals Through Linear Systems

- The impulse response of a system is defined as the response of the system (with zero initial conditions) to a unit impulse (or delta function $\delta(t)$) applied to the input of the system.



$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

This relation is called the *convolution integral*.

Causality and Stability

- Causality

$$h(t) = 0, \quad t < 0$$

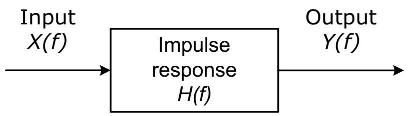
- Bounded Input-Bounded Output (BIBO) stability criterion

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Frequency Response

- Define the transfer function or frequency response of the system as the Fourier transform of its impulse response

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t}\, dt$$



$$Y(f) = H(f)X(f)$$

Frequency Response II

where $H(f) = |H(f)|e^{j\beta(f)}$

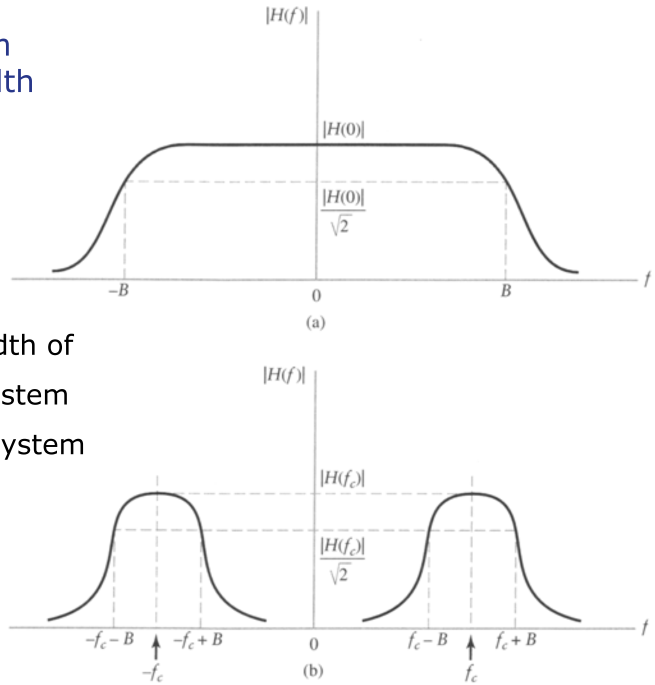
properties: $|H(f)| = |H(-f)|$

$$\beta(f) = -\beta(-f)$$

gain in decibels (dB):

$$\alpha'(f) = 20\,\log_{10}|H(f)|$$

System Bandwidth



System bandwidth of

(a) Low-pass system

(b) Band-pass system

Correlation and Spectral Density: Energy Signals

- Autocorrelation function of energy signal *x(t)*

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\, dt$$

- The energy of the signal *x(t)* is

$$R_x(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

- The energy spectral density or energy density spectrum of an energy signal *x(t)* is

$$\psi_x(f) = |X(f)|^2$$

Correlation and Spectral Density: Energy Signals II

- Wiener-Khintchine Relations for energy signals
  - The autocorrelation function $R_x(\tau)$ and energy spectral density $\psi_x(f)$ for a Fourier-transform pair.

$$\psi_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_x(\tau) = \int_{-\infty}^{\infty} \psi_x(f) e^{j2\pi f\tau} df$$

Note that the Fourier transformation is performed with respect to the adjustable lag τ.

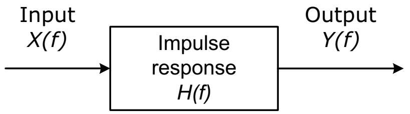
Correlation and Spectral Density: Energy Signals III

- Some relations

$$\int_{-\infty}^{\infty} R_x(\tau) d\tau = \psi_x(0)$$

$$\int_{-\infty}^{\infty} \psi_x(f) df = R_x(0)$$

Effect of Filtering on Energy Spectral Density



$$Y(f) = H(f)X(f)$$

$$\psi_y(f) = |H(f)|^2 \psi_x(f)$$

where

$$\psi_x(f) = |X(f)|^2$$

$$\psi_y(f) = |Y(f)|^2$$

Cross-correlation of Energy Signals

- The autocorrelation function provides a measure of the similarity between a signal and its own time-delayed version.
- The *cross-correlation function* provides a measure of the similarity between one signal and the time-delayed version of a second signal.
- The cross-correlation between *x(t)* and *y(t)* is:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t-\tau) dt$$

Cross-correlation of Energy Signals II

- The two signals *x(t)* and *y(t)* are somewhat similar if their cross-correlation function $R_{xy}(\tau)$ is finite over some range of τ.
- They are said to be *orthogonal* over the entire time interval if

$$R_{xy}(0) = 0$$

that is if

$$\int_{-\infty}^{\infty} x(t) y^*(t)\, dt = 0$$

Cross-correlation of Energy Signals III

- An important property

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

- Cross-spectral density is defined as

$$\psi_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{(-j2\pi f\tau)}\, d\tau$$

or

$$\psi_{xy}(f) = X(f) Y^*(f)$$

Power Spectral Density (PSD)

- The *average power* of a signal *x(t)* is defined by

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

- The signal *x(t)* is said to be a power signal if the condition

$$P < \infty$$

holds.

- Define

$$x_T(t) = x(t) rect\left(\frac{t}{2T}\right)$$
$$= \begin{cases} x(t), & -T \le t \le T \\ 0, & otherwise \end{cases}$$

Power Spectral Density (PSD) II

- The *average power* in terms of $x_T(t)$ is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

- The *power spectral density* or *power spectrum* of the power signal *x(t)* is

$$S_x(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$$

where

$$\frac{1}{2T} |X_T(f)|^2$$

is called the periodogram of the signal.

Power Spectral Density (PSD) III

- The *average power* is

$$P = \int_{-\infty}^{\infty} S_x(f)\,df$$

- Therefore, the total area under the curve of the power spectral density of a power signal is equal to the average power of that signal.