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ase note that you have 4 questions and 8 pages.

5%) Solve the IBVP:

DE: $u_t + x u_x = x$ $x>0,\ t>0$

C: $u(0,t)=0$ $t>0$

: $u(x,0)=0$ $x>0$

$U(x,s) = \mathcal{L}(u(x,t))(s) = \int_0^\infty u(x,s)\, e^{-st}\, dt$

$\mathcal{L}(u_t) = s\,U(x,s) - \cancel{u(x,0)}^{\,0}$

$\mathcal{L}(x u_x) = x\,U_x(x,s)$

$\mathcal{L}(x) = x\,\mathcal{L}(1) = \frac{x}{s}$

$+ x U_x = \frac{x}{s} \Rightarrow U_x + \frac{s}{x} U = \frac{1}{s}$ First order - ODE.

$U(x,s) = A e^{-\int \frac{s}{x} dx} + e^{-\int \frac{s}{x} dx} \int \frac{1}{s} e^{\int \frac{s}{x} dx} dx$

$= A e^{-s\ln x} + e^{-s\ln x} \int \frac{1}{s} e^{s\ln x} dx$

$= \frac{A}{x^s} + \frac{1}{x^s} \int \frac{x^s}{s} dx = \frac{A}{x^s} + \frac{1}{x^s} \frac{x^{s+1}}{s(s+1)}$

$= \frac{A}{x^s} + \frac{x}{s(s+1)}$

$\mathcal{L}(u(0,t))(s) = U(0,s) = 0 = \frac{A}{0} + 0 \Longrightarrow$ therefore A must be zero

$U(x,s) = \frac{x}{s(s+1)} = x\left(\frac{1}{s} - \frac{1}{s+1}\right)$

$s))(t) = x\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) - x\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

$\boxed{(x,t) = x\left[1 - e^{-t}\right]}$

$e^{-\int \frac{s}{x} dx} \left[\int_0^\infty \frac{1}{s} e^{\int \frac{s}{x} dx} dx + c \right]$

2) (25%) Consider the 1-d wave equation:

PDE: $u_{tt} = u_{xx} \quad -\infty < x < \infty, \quad t > 0$

ICs: $\begin{cases} u(x,0) = f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \\ u_t(x,0) = g(x) = 0. \end{cases}$

D'alembert formula directly.

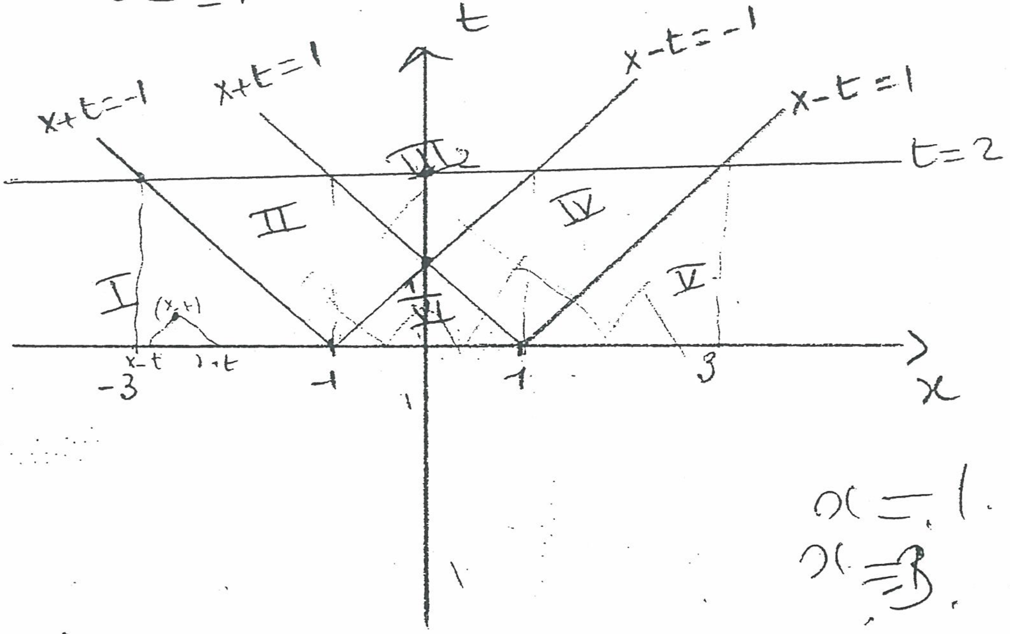
a) Find the solution $u(x,t)$ in the different regions of the xt-plane.

b) Sketch, in particular, the solution for $t = 2$.

a) D'Alembert's Formula: for $\boxed{c=1}$

$$u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \cancel{\frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz} \to 0$$

$x \pm ct = \pm 1$

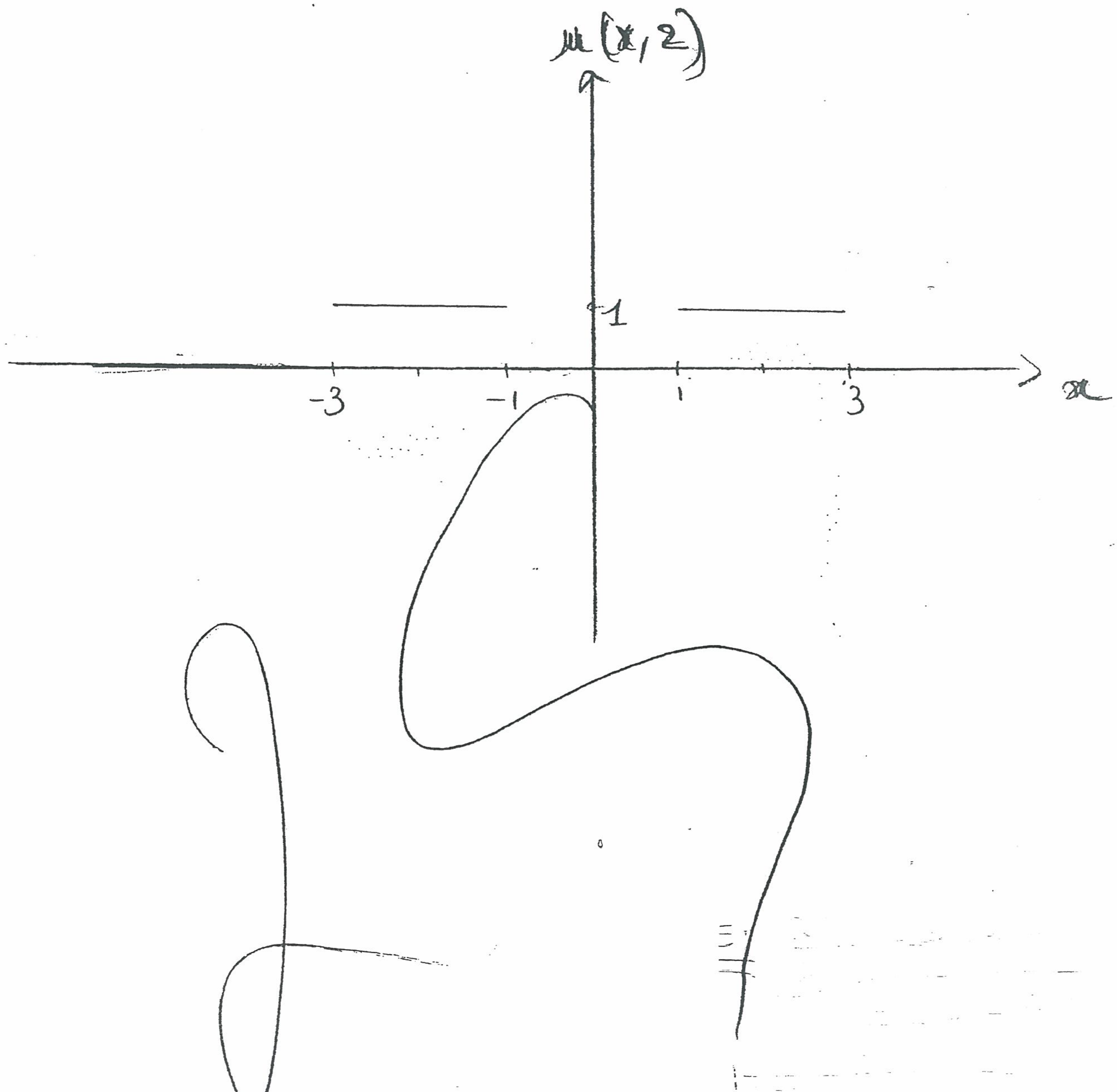


$x=1.$ $x=1.$
$x=3.$ $x=3.$

$$u(x,t) = \begin{cases} \frac{1}{2}[0+0] = 0 & (x,t) \in \mathrm{I} \\ \frac{1}{2}[0+2] = 1 & (x,t) \in \mathrm{II} \\ \frac{1}{2}(0+0) = 0 & (x,t) \in \mathrm{III} \\ \frac{1}{2}[2+0] = 1 & (x,t) \in \mathrm{IV} \\ \frac{1}{2}[0+0] = 0, & (x,t) \in \mathrm{V} \\ \frac{1}{2}[2+2] = 2 & (x,t) \in \mathrm{VI} \end{cases}$$

b) $$u(x,2) = \begin{cases} 0 & -\infty < x < -3 \\ 1 & -3 < x < -1 \\ 0 & -1 < x < 1 \\ 1 & 1 < x < 3 \\ 0 & 3 < x < \infty \end{cases}$$

THE DEBATE CLUB



3) (25%)

a) Use the method of characteristics to solve the IVP:

PDE: $xu_x + u_t + tu = 0 \qquad -\infty < x < \infty, \quad t > 0$

IC: $u(x,0) = f(x) \qquad -\infty < x < \infty$

b) Plot few characteristics in the xt-plane, and explain the relationship between them and the PDE.

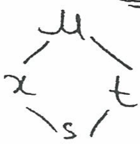
c) ~~Sketch the solution u versus x for $t = 1$.~~

a) . Reduce PDE to ODE:

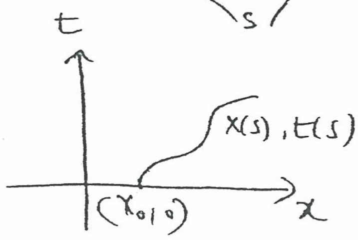
Find: χ_curves $(x(s), t(s))$

Along χ_curves.

Chain rule.



$$\frac{du}{ds} = u_x \frac{dx}{ds} + u_t \frac{dt}{ds}$$



$\frac{dx}{ds} = x \qquad x(0) = x_0$

$\frac{dt}{ds} = 1 \qquad t(0) = 0 \qquad$ 2 ODE's

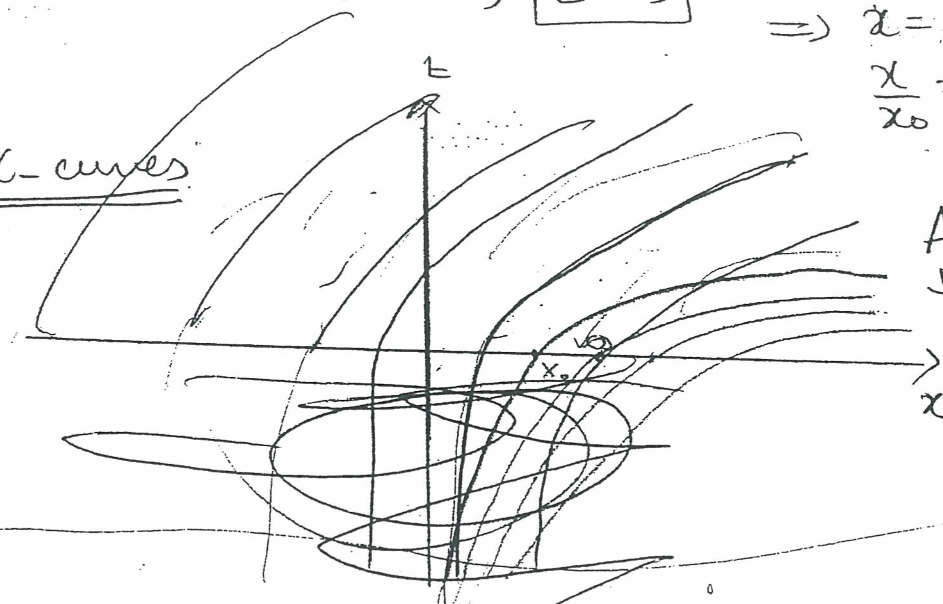
$\frac{dx}{x} = ds \qquad \ln\frac{x}{c} = s \Rightarrow x = ce^s \overset{IC}{\Rightarrow} \boxed{x = x_0 e^s}$

$dt = ds \qquad t = s + c \overset{IC}{\Rightarrow} \boxed{t = s}$

$\Rightarrow x = x_0 e^t$

$\frac{x}{x_0} = e^t \Rightarrow \boxed{t = \ln\frac{x}{x_0}}$

b) χ_curves



Along these χ_curves the PDE will be reduced to an ODE

$$\boxed{\frac{du}{ds} + s\,u = 0}$$

$\frac{du}{ds} = -su \Rightarrow \frac{du}{u} = -s\,ds$

$\ln\frac{u}{c} = -\frac{s^2}{2} \Rightarrow u_{(x_0,s)} = c\,e^{-\frac{s^2}{2}}$

$\boxed{IC: u(x_0, 0) = f(x_0)} = c \Rightarrow$ 5/8 $u(x_0, s) = f(x_0)\, e^{-\frac{s^2}{2}}$

$$u(x_0, s) = f(x_0)\, e^{-\frac{s^2}{2}}$$

$$x_0 = x e^{-t}$$

$$s = \ln \frac{x}{x_0} = \ln \frac{x}{x e^{-t}} = \ln e^{t} = t$$

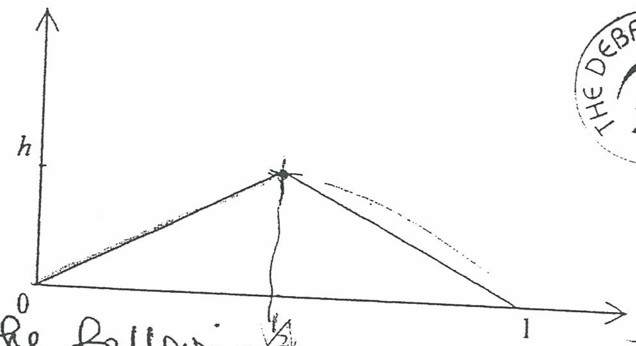
$$\boxed{u(x,t) = f(x e^{-t})\, e^{-\frac{t^2}{2}}}$$

c) sketch the solution u versus x for $t = 1$

$$u(x,1) = f\left(\frac{x}{e}\right) e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}\, f\left(\frac{1}{e} \cdot x\right)$$

it depends on the shape of $f(x)$ which is not given.

4) (25%) A guitar string of length $L = 1$ is pulled upward at the middle so that it reaches height h (see figure). What is the subsequent motion of the string if it is suddenly released?



The Problem is the following

PDE: $u_{tt} = c^2 u_{xx}$ $\quad 0<x<1 \quad t>0$.

BC: $u(0,t)=0$, $u(1,t)=0$ $\quad t>0$

IC: $u(x,0) = f(x) = \begin{cases} 2hx & 0<x<\frac{1}{2} \\ 2h(1-x) & \frac{1}{2}<x<1 \end{cases}$

$u_t(x,0) = g(x) = 0$

suddenly released $\Rightarrow$ $g(x)=0$

$f(x)$ is found as follow

① $(0,0)$ $\quad (\frac{1}{2}, h)$

$y = ax$

$h = \frac{a}{2}$

$\boxed{y = 2hx}$

② $(1,0)$, $(\frac{1}{2}, h)$

$y = ax+b$

$0 = a+b$

$h = \frac{1}{2}a + b$

$-h = \frac{1}{2}a$

$\boxed{a = -2h}$

$b = 2h$

$y = -2hx + 2h$

$= 2h(1-x)$

The solution of the finite guitar problem is the following:

$$u(x,t) = \sum_1 \left(a_n \sin n\pi\alpha t + b_n \cos n\pi\alpha t\right) \sin n\pi x$$

where $a_n = \frac{2}{n\pi\alpha} \int_0^1 g(x) \sin n\pi x \, dx = 0$ $\quad (g(x)=0$.

& $b_n = 2\int_0^1 f(x) \sin n\pi x \, dx$

$$= 2\left[\int_0^{1/2} 2hx \sin n\pi x \, dx + \int_{1/2}^1 2h(1-x) \sin n\pi x \, dx\right]$$

$$= 2\left[\left(-2hx \frac{\cos n\pi x}{n\pi} + 2h\frac{\sin n\pi x}{(n\pi)^2}\right)_0^{1/2} + \left(-2h(1-x)\cdot\frac{\cos n\pi x}{n\pi} - 2h\frac{\sin n\pi x}{(n\pi)^2}\right)_{1/2}^1\right.$$

$$= 2\left[\cancel{-h\frac{\cos n\pi/2}{n\pi}} + \frac{2h \sin \frac{n\pi}{2}}{(n\pi)^2}\right] + 2\left[\cancel{+h\frac{\cos \frac{n\pi}{2}}{n\pi}} + \frac{2h \sin\frac{n\pi}{2}}{(n\pi)^2}\right.$$

$$= \frac{8h \sin\frac{n\pi}{2}}{(n\pi)^2}$$

int

x	$\sin n\pi x$
1	$-\frac{\cos n\pi x}{n\pi}$
0	$-\frac{\sin n\pi x}{(n\pi)^2}$

$(1-x)$	$\sin n\pi x$
-1	$-\frac{\cos n\pi x}{n\pi}$
0	$-\frac{\sin n\pi x}{(n\pi)^2}$

$\frac{d^2U}{du^2} \ldots s^2 U(x,s) + s\,u(x,0) + u_t(x,0) = \frac{\sin \pi x}{s}$

$m - s^2$

$c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} - \frac{\sin \pi x}{\cancel{s^3}}$

$c_1 = 0.$

$c_2 e^{-\sqrt{s}x} - \frac{\sin \pi x}{s^3}$

$U(0,s) = \mathcal{L}\, u(0,t)$

$-\frac{\sin \pi x}{s^3}$

for $n=1$ $\quad b_1 = \frac{8h}{\pi^2}$

$n=2$ $\quad b_2 = 0$

$n=3$ $\quad b_3 = \frac{-8h}{9\pi^2}$

$n=4$ $\quad b_4 = 0$

$n=5$ $\quad b_5 = \frac{8h}{25\pi^2}$

$$u(x,t) = \frac{8h}{\pi^2}\cos \pi \alpha t \cdot \sin \pi x + 0 - \frac{8h}{9\pi^2}\cos 3\pi\alpha t \cdot \sin 3\pi x$$
$$+ 0 + \frac{8h}{25\pi^2}\cos 5\pi\alpha t \cdot \sin 5\pi x + 0 + \cdots$$

$$\frac{dU_{(w,t)}}{dt} = \alpha^2 w^2 U_{(w,t)} - B\, U_{(w,t)}.$$

$U_{(w,0)}$.

$F(u(x,0)) = \alpha(w)$.

$$sU_{(x,s)} - \cancel{u}(x,0) + x\frac{dU_{(x,s)}}{dx}$$

$$= \frac{x^2}{s},$$

$$\frac{dU}{-U_{(w,t)}(\alpha^2 w^2 + B)} =$$

$$U_{(0,s)} = 0$$

$$U_{(w,t)} = \phi(w)\, e^{-(\alpha^2 w^2 + B)t}$$

$$\frac{dU_{(x,s)}}{dx} + \frac{s}{x}U_{(x,s)} = $$

$$x \frac{1}{\sqrt{2\pi}}\int_0^\infty u(x)\, e^{i w x}\, e^{-w^2 \alpha^2 t}\, dw.$$

$$c_1 e^{-\int \frac{s}{x}dx}\int \frac{x}{s} e^{\int \frac{s}{x}dx} dx$$

let $\frac{dU}{dt} =$

$$2u = c_1 e^{-s\ln x}$$

$$du = c_1 x^{-s}$$

$$\frac{c_1}{x^s} \times \int \frac{x}{s}\, x^s\, dx$$

$$\frac{x^{s+2}}{s(s+2)} \quad \frac{x^{s+1}}{s}.$$

$$e^{-\int \frac{s}{x}dx}\int$$