1) (25%) Let $u(x,t)$ denote the concentration (at time t, at position x) in a moving medium (moving from left to right with speed $v = 1$) where the concentration at the ends of the medium are kept at 0 (by some filtering device), and the initial concentration is $e^{x/2}$. The corresponding IBVP is:

PDE: $u_t = u_{xx} - u_x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = 0 \qquad 0 < t < \infty$

IC: $u(x,0) = e^{x/2} \qquad 0 \le x \le 1$

a) Transform the above problem in u to a new problem in w, where $u(x,t) = e^{\frac{x}{2} - \frac{t}{4}} \cdot w(x,t)$.

b) Solve the resulting IBVP in w by the method of separation of variables. Show all details.

c) Deduce the solution u of the given IBVP, and find the steady-state solution $u(x,\infty)$.

a)

$$u_t = \left(-\frac{1}{4} e^{\frac{x}{2}-\frac{t}{4}}\right) w(x,t) + e^{\frac{x}{2}-\frac{t}{4}} w_t(x,t)$$

$$u_x = \frac{1}{2} e^{\frac{x}{2}-t/4} w(x,t) + e^{\frac{x}{2}-\frac{t}{4}} w_x(x,t)$$

$$u_{xx} = \frac{1}{4} e^{\frac{x}{2}-t/4} w(x,t) + \frac{1}{2} e^{x/2 - t/4} w_x(x,t) + \frac{1}{2} e^{\frac{x}{2}-t/4} w_x{}' + e^{\frac{x}{2}-t/4} w_x$$

Subs in P.D.E

$$\rightarrow -\frac{1}{4} e^{\frac{x}{2}-t/4} w(x,t) + e^{x/2 - t/4} w_t(x,t)$$

$$= \frac{1}{4} e^{x/2 - t/4} w(x,t) + \frac{1}{2} e^{x/2 - t/4} \cancel{w_x(x,t)} + e^{x/2 - t/4} w$$

$$- \frac{1}{2} e^{x/2 - t/4} w(x,t) - e^{x/2 - t/4} \cancel{w_x(x,t)}$$

$$\rightarrow \cancel{e^{x/2 - t/4}} w_t(x,t) = \cancel{e^{x/2 - t/4}} w_{xx}(x,t)$$

$$\rightarrow \text{PDE} \quad w_t(x,t) = w_{xx}(x,t)$$

wrt 1st B.C

$$u(0,t) = e^{-t/4} \cdot w(0,t) = 0 \longrightarrow w(0,t) = 0$$

the new Pb

$$w_t(x,t) = w_{xx}(x,t)$$

$$w(0,t) = 0$$
$$w(1,t) = 0$$
$$w(x,0) = 1$$

1) sep of var. $w(x,t) = X(x)\,T(t)$

$$w_t(x,t) = X(x)\,T'(t)$$
$$w_{xx}(x,t) = X''(x)\,T(t)$$

$$\rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = k = -\lambda^2$$

choosing k to be negative because as $t \rightarrow \infty$ for W not to blowup

$$\frac{T'(t)}{T(t)} = -\lambda^2 \quad \text{... 1st order O}$$

$$\rightarrow \quad T(t) = A_1 e^{-\lambda^2 t}$$

$$\frac{X''(x)}{\lambda(x)} = -\lambda^2$$

$$\rightarrow \quad X(x) = B\cos\lambda x + A\sin\lambda$$

$$\rightarrow \quad w(x,t) = D\,e^{-\lambda^2 t}\left[B\cos\lambda x + A s\right.$$

using first B.C $\quad w(0,t) = 0 \rightarrow e^{-\lambda^2 t} A B ~~\cos\lambda 0~~ = 0$

$$\longrightarrow \quad B = 0$$

$$\rightarrow \quad w(x,t) = A e^{-\lambda^2 t} \sin\lambda x$$

2nd BC $\quad \rightarrow \quad w(1,t) = A e^{-\lambda^2 t}\sin\lambda = 0$

if $A = 0$ then $w(x,t) = 0$ is the tri

$\rightarrow A \neq 0 \rightarrow \sin\lambda = 0 \quad \lambda_n = n\pi$

$$\rightarrow \quad w_n(x,t) = A_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

due to super position principle

$\sum^{\infty}$ [illegible] $^{-(n\pi)^2 t}$ [illegible]

$$\rightarrow \quad A_1 \sin \pi x + A_2 \sin 2\pi x + \dots + A_n \sin n\pi x = 1$$

ⓧ by $\sin m\pi x$ $\int_0^1$ $\longrightarrow$ $A_1\int_0^1 \sin \pi x \sin m\pi x + A_2 \int_0^1 \sin 2\pi x \sin m\pi x$

$$+ A_n \int_0^1 \sin n\pi x \sin m\pi x$$

$$= \int_0^1 \sin m\pi x$$

considering $m = n$

$$\rightarrow \quad \frac{A_n}{2} = \int_0^1 \sin n\pi x$$

$$\rightarrow \quad A_n = 2\int_0^1 \sin n\pi x$$

$$= \frac{2}{n\pi}\left(-\cos n\pi x\right)_0$$

$$= \frac{2}{n\pi}\left(-\cos n\pi + 1\right)$$

$$= \frac{2}{n\pi}\left(-(-1)^n + 1\right)$$

$$A_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

so sol. $w(x,t) = \sum_1^\infty A_n e^{-(n\pi)^2 t} \sin n\pi$

but $u(x,t) = e^{x/2 - \frac{t}{4}} \sum_1^\infty A_n e^{-(n\pi)^2 t} \sin$

$$= \sum_1^\infty A_n e^{\frac{x}{2} - \frac{t}{4} - (n\pi)^2 t} \sin n\pi$$

$$= \sum_1^\infty A_n e^{\frac{x}{2} - \left(\frac{1}{4} + (n\pi)^2\right)t} \sin$$

as $t \to \infty$ $\quad e^{\frac{x}{2} - \left(\frac{1}{4} + (n\pi)^2\right)t} \longrightarrow 0$

$$\longrightarrow \quad u(x,\infty) \rightarrow 0$$

② (20%) Use the Laplace Transform on t to solve the IBVP:

THE DEBATE CLUB

PDE: $u_t + xu_x = x \qquad 0 < x < \infty, \qquad 0 < t < \infty$

BC: $u(0,t) = 0 \qquad 0 < t < \infty$

IC: $u(x,0) = 0 \qquad 0 \le x < \infty$

Given: $L\left(e^{at}\right)(s) = \frac{1}{s-a}$ for $s > a \qquad (a \in \mathbb{R})$

let $U(x,s) = \mathcal{L}(u(x,t)) = \int_0^\infty u(x,t)$

$\mathcal{L}(u_t)(s) = sU(x,s) - u(x,0) = sU$

$\mathcal{L}(u_x)(s) = U_x(x,s)$.

$\mathcal{L}(x)(s) = \frac{x}{s}$.

transforming into the PDE

$sU + xU_x = \frac{x}{s}$. get the final ODE.

$xU_x + sU = x. \Rightarrow U + \cancel{\frac{x}{s}U_x} = \quad U_x + \frac{s}{x}U = \frac{1}{s}$

$\Rightarrow \cancel{U_x + \frac{s}{x}U = x}$. $\qquad xU_x + sU = \frac{x}{s} \Rightarrow \boxed{U_x + \frac{s}{x}U = :}$

the solution is given by $U = Ae^{-\int a(x).dx} + C(x)e^{-\int a(x).dx}$.

$C(x) = \int b(x)\, e^{\int a(x).dx}.dx$.

$\int a(x).dx = \int \frac{s}{x}.dx = s\ln|x| = \ln x$

$a(x) = \frac{s}{x} \quad b(x) = x$.

$e^{\int a(x).dx} = x^s$

$U(x,s) = Ae^{-\ln(x)^s} + \frac{x^s.x^2}{s+2}.x^s$

$C(x) = \int \frac{1}{s}.x^s dx = \frac{x^{s+1}}{s+1}$

~~Use BCs / u(0,t) = 0.~~ $\Rightarrow U(x,s) = Ax^{-s} + \frac{x^{s+1}}{(s+1)}.x^{-s} = \frac{A}{\quad}$

Now use BC

$$\Rightarrow u(0,s) = 0 = \frac{A}{0^5} \Rightarrow A = 0 .$$

$\frac{A}{s} + \frac{B}{s+1}$

$$\Rightarrow U(x,s) = \frac{x}{(s+1).s}$$

$$\Rightarrow u(x,t) = \mathcal{L}^{-1}(U_{x,s}) = \mathcal{L}^{-1}\left(\frac{x}{s}\cdot\frac{1}{s+1}\right) \rightarrow \frac{K}{s} + \frac{K}{s+1}$$

$\frac{A}{s} + \frac{Bx+C}{s+1}$

6

$$\mathcal{L}^{-1} = \mathcal{L}^{-1}\left(\frac{x}{s}\right) * \mathcal{L}\left(\frac{1}{s+1}\right)$$

$$= x * e^{-t}$$

$A(s+B) + B(s)$

$A(s+1) + Bx s + C$

$$= \int_0^\infty x . e^{-(t-\tau)} d\tau$$

$Bx s = 0$

$A s+1 + C$

$x(1-\tau) e^{-\tau} d\tau$

$$\int_0^\infty x e^{-\tau} d\tau - x\tau e^{-\tau} = \int_0^\infty x . e^{-t} . e^{\tau} d\tau \Rightarrow x . e^{-t}\left[e^{\tau}\right]_0^\infty = -xe^{-t} .$$

$\frac{x}{s} - \frac{x}{s+1} .$

$x\, e^{-(t-\tau)} d\tau$

$x e^{-t} e^{\tau} d\tau$

$\int_0^\infty ($

$A(s+1) + B$

$x - xe^{-t}$

$$\int_0^\infty (x-\tau) e^{-\tau} d\tau$$

$x(1-e^{-t})$

$$\Rightarrow \int_0^\infty x e^{-t} d\tau - \int_0^\infty \tau e^{-\tau} d\tau$$

$e^{-\tau} d\tau$

$\Big|^\infty$

$B = 0$

$A = x$

3) (20%) Solve the IVP:

PDE: $u_{xx} + 3u_{xy} = 0 \qquad 0 < x < \infty, \qquad -\infty < y < \infty$

ICs: $u(0, y) = f(y) \qquad -\infty < y < \infty$

$u_x(0, y) = g(y) \qquad 0 < x < \infty, \qquad -\infty < y < \infty$

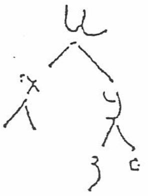
Hint: Let $\xi = y, \quad \eta = y - 3x$.

$\xi = y$

$\eta = y - 3x$

$3x = y - \eta$

$= \xi - \eta \Rightarrow x = \frac{\xi}{3} - \frac{\eta}{3}$



$u_x = u_\xi \, x_\xi + u_\eta . x_\eta$

$= u_\xi . \frac{1}{3} + u_\eta \frac{1}{3}$

$u_{xx} = \left(\frac{u_\xi}{3}\right)_x \mp \left(\frac{u_\eta}{3}\right)_x =$

$u_{xy} = \left(\frac{u_\xi}{3}\right)_y \mp \left(\frac{u_\eta}{3}\right)_y = \frac{1}{3} u$

ξ

$\eta_y = 1 \qquad \eta_x = -3$

$\xi_y = 1 \qquad \xi_x = 0$

$u_x = \underbrace{u_\xi . \xi_x}_{0} + u_\eta . \eta_x$

$u_x = -3 u_\eta$

$u_{xx} = -3 u_{\eta\eta}$

$u_{xy} = -3 (u_\eta)_y = -3 \left(u_\eta . [u_\xi . \xi_y + \xi_\eta . \right.$

$= -3 u_{\eta\xi} + \xi_{\eta\eta}$

$\Rightarrow 3u_{\eta\eta} \; 3u_{\eta\xi} + 3_{\eta\xi}$

$\Rightarrow 3 u_{\eta\xi} = 0$

$u_{\eta\xi} = 0$

$\Rightarrow u(x,t) = \phi$

Using Laplace by x.

$\mathcal{L}(u_{xx}) = s^2 \mathcal{L} U - s U(0,y) - U_x(0,y)$.

$\mathcal{L} U_{xy}) = s U - U(0,y) \Rightarrow s U_y - f(y)$.

$\Rightarrow s^2 U - s f(y) - g(y) + 3 s U_y - 3 f(y) = 0$

$3 s U_y + s^2 U - s f(y - y(y)) - 3 f(y) = 0$

$\Rightarrow u(x,y) = \phi(y) + \psi(y-3x)$.

Now use IC's.

$u(0,y) = \phi(y) + \psi(y) = f(y)$ ①

$u_x(0,y) = -3\psi' = g(y)$ ②.

To solve

get $\psi = -\frac{1}{3}\int_{x_0}^{x} g(\tau)\,d\tau + \psi(x_0)$

$\phi(y) = f(y) - \psi(y)$. ?

$=$

12

4) (20%) a) Write down, without proof, D'Alembert's formula for the solution of the 1-d wave equation:

PDE: $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty, \quad 0 < t < \infty$

ICs: $\begin{cases} u(x,0) = f(x) & -\infty < x < \infty \\ u_t(x,0) = g(x) & -\infty < x < \infty \end{cases}$

b) Let $c = 1$, $\quad f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$, $\quad g(x) = 0$.

(i) Find the solution $u(x,t)$ in the different regions of the xt-plane.

(ii) Sketch the solution for $t = 0$ and for $t = 1$.

a) $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty$, $0 < t < \infty$.

IC's : $\begin{cases} u(x,0) = f(x) & -\infty < x < \infty \\ u_t(x,0) = g(x) & -\infty < x < \infty . \end{cases}$

It is an infinite string problem of a 1.D. wave. the solution is given by. D'Alembert formula.

$$u(x,t) = \frac{1}{2}\left[(f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-t}^{x+ct} g(\xi) dT.$$

b) let $c=1$ ..

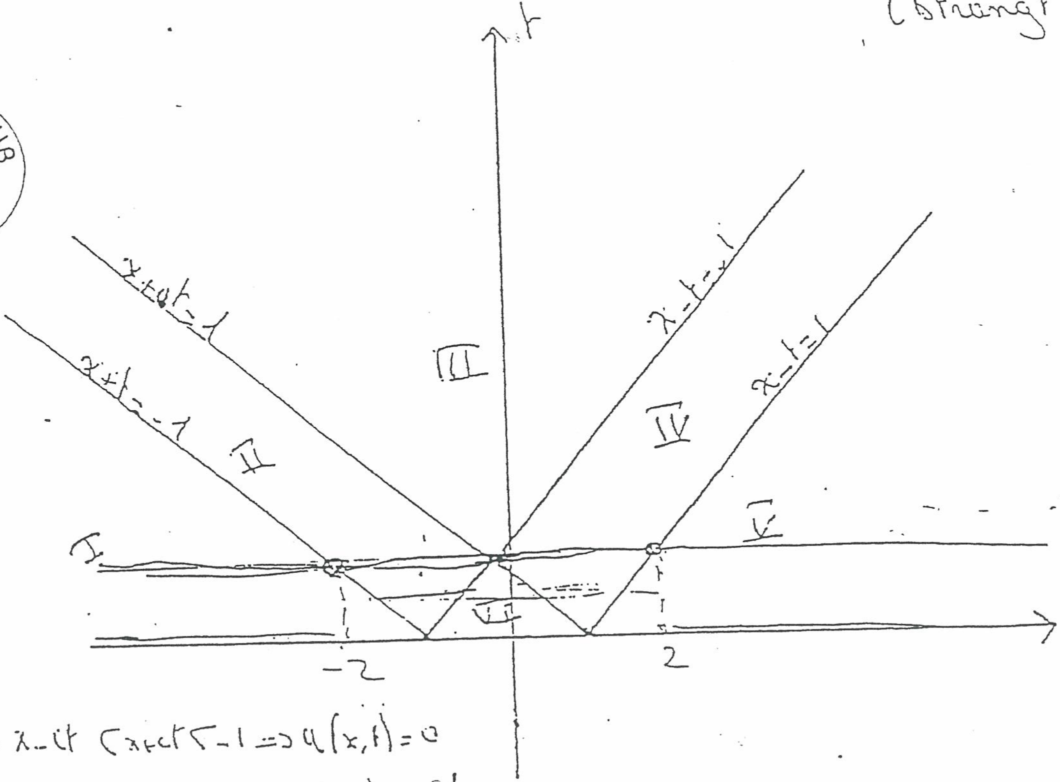
$u_{tt} = u_{xx}$.

$u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

$u_t(x,0) = 0$.

i) By applying the D'Alembert formula to the given probl the following problem can be solved by sketching the region in xt plane we have $x \pm ct = \pm 1$.

(straight lines of --



$(x,t) \in I \Rightarrow x-ct < x+ct < -1 \Rightarrow u(x,t)=0$

$(x,t) \in II \Rightarrow x-ct < -1 < x+ct < 1 \Rightarrow u(x,t)=\frac{1}{2}$.

$(x,t) \in III \Rightarrow x-ct < -1;\ x+ct > 1 \Rightarrow u(x,t)=0$

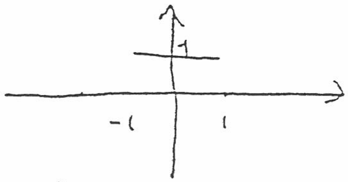
$x,t) \in IV \Rightarrow -1 < x-ct < 1;\ x+ct > 1 \Rightarrow u(x,t)=1/2$.

$x,t) \in V \Rightarrow u(x,t)=0$

$x,t) \in VI \Rightarrow u(x,t)=1$

$\therefore$) for $t=0$; $u(x,t)=\frac{1}{2}[f(x)+f(x)]=f(x)$.

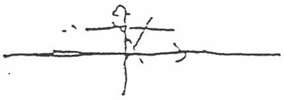
$\Rightarrow$



→ shifting 1 unit right

→ shifting 1 unit left.

for $t=1 \Rightarrow u(x,t)=\frac{1}{2}[f(x+1)+f(x-1)]$.



Travelling Wave