NDU

MAT 335

Partial Differential Equations

Exam# 1

Tuesday April 19, 2005

Duration: 60 minutes

Name: [redacted]

ID#: [redacted]

Section: MWF 9 → 10

Grade: 94

THE DEBATE CLUB

(25%)

a) Use the Fourier transform to solve the IVP:

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a)

Let $U(\xi,t) = \mathcal{F}(u(x,t))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\xi x}\, dx$

PDE

$\Rightarrow U_t(\xi,t) = -\alpha^2 \xi^2 U(\xi,t) - \beta U(\xi,t)$

$U_t(\xi,t) + (\alpha^2 \xi^2 \beta) U(\xi,t) = 0$ ODE

$U_t(\xi,t) = C e^{-(\alpha^2 \xi^2 \beta)t}$

IC: $U(\xi,0) = C = \Phi(t) \quad \left(\Phi = \mathcal{F}\, \phi(x) \right.$

$U(\xi,t) = \Phi(t) \cdot e^{-(\alpha^2 t)\xi^2} \cdot e^{-\beta t}$

$u(x,t) = \mathcal{F}^{-1}(U(\xi,t)(x)) = \mathcal{F}^{-1}\Phi(t) * \mathcal{F}^{-1}(e^{-(\alpha^2 t}$

$= e^{-\beta t} \phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right)$

$\mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right) = \mathcal{F}^{-1}\left(e^{-\xi^2/4a^2}\right) = e^{-a^2 x^2} \cdot a\sqrt{2}$

$\frac{1}{4a^2} = \alpha^2 t \Rightarrow a = \frac{1}{2\alpha\sqrt{t}}$

$$\Rightarrow u(x,t) = e^{-\beta t} \phi(x) * \frac{\sqrt{2}}{2\alpha\sqrt{t}} e^{-\frac{x^2}{4\alpha^2 t}}$$

$$= \frac{e^{-\beta t}}{\sqrt{2t}\,\alpha} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x-z)\, e^{-\frac{z^2}{4\alpha^2 t}}\, dz$$

$$\boxed{u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{t\pi}} \int_{-\infty}^{\infty} \phi(x-z)\, e^{-\frac{z^2}{4\alpha^2 t}}\, dz}$$

b) for $\phi(x) = 1$

$$u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{t\pi}} \int_{-\infty}^{\infty} 1 \cdot e^{-\frac{z^2}{4\alpha^2 t}}\, dz$$

let $u = \frac{z}{2\alpha\sqrt{t}}$

$$\Rightarrow du = \frac{1}{2\alpha\sqrt{t}} dz \Rightarrow dz = du\,(2\alpha\sqrt{t})$$

$$\Rightarrow u(x,t) = \frac{e^{-\beta t}\,\cancel{2\alpha\sqrt{t}}}{\cancel{2\alpha\sqrt{t}}\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{\sqrt{\pi}}$$

$$= e^{-\beta t} \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}}$$

$$= \boxed{e^{-\beta t}}$$

4) (10%) Let $u = u(x, y, z)$ be a function of 3 independent variables possessing derivatives of all orders. Solve the (possibly easiest) 3rd order PDE:

$u_{xyz} = 0, \qquad x, y, z \in \mathbb{R}.$

$$u = u(x, y, z) \qquad u_{xyz} = 0$$

$$\int u_{xyz}\, dx = \int 0\, dx$$

$$u_{yz} = f(y, z)$$

$$\int u_{yz}\, dy = \int f(y, z)\, dy$$

$$u_z = f(y, z) + f(x, z)$$

$$\int u_z\, dz = \int \left(f(y, z) + f(x, z) \right) dz$$

$$u = f_1(y, z) + f_2(x, z) + f_3(x, y)$$

where f_1, f_2, f_3 are all deferentibal equations

$\Rightarrow \cancel{X = cK} \Rightarrow \cancel{X = cst}$

$X' = K$

$\Rightarrow \boxed{X(x) = x}$

$$u(x,t) = X(x)\,T(t) = x\left(\frac{x+1}{Kt(x+1)+1}\right) \qquad K = cst$$