1) (25%) Use the Laplace Transform to solve the IBVP:

PDE: $u_t + xu_x = x^2$ $\quad 0 < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = 0$ $\quad 0 < x < \infty$

BC: $u(0,t) = 0$ $\quad 0 < t < \infty$

$$\mathcal{L}\big(u(x,t)\big) = U(x,s)$$

then $\mathcal{L}(P.D.E)$,

$$\Rightarrow \quad sU(x,s) - \cancel{u(x,0)}^{\,0} + xU_x(x,s) = \frac{x^2}{s}$$

$$\Rightarrow \quad \boxed{U_x + \frac{s}{x}U = \frac{x}{s}}$$

from formula (4) of the O.D.E formula sheet

$$U(x,s) = Ae^{-\int \frac{s}{x}dx} + c(x)\, e^{-\int \frac{s}{x}dx}$$

where $c(x) = \int \frac{x}{s} e^{\int \frac{s}{x}dx}\, dx$

$$= \int \frac{x}{s} e^{\ln x^s}\, dx$$

$$= \int \frac{x \cdot x^s}{s} dx = \int \frac{x^{s+1}}{s} dx$$

$$= \frac{x^{s+2}}{s(s+2)}$$

then $U(x,s) = A\, e^{\ln \frac{1}{x^s}} + \frac{x^{s+2}}{s(s+2)}\, e^{\ln\left(\frac{1}{x^s}\right)}$

$$= \frac{A}{x^s} + \frac{x^{s+2}}{s(s+2)} \cdot \frac{1}{x^s} = \frac{A}{x^s} + \frac{x^2}{s(s+2)}$$

$$U(x,s) = \frac{A}{x^s} + \frac{x^2}{s(s+2)}$$

$$\mathcal{L}(B.C) = \mathcal{L}\left(u(0,t)\right) = U(0,s) = 0$$

then $U(0,s) = 0$ applies that A must be zero, since $U(x,s)$ and hence $u(x,t)$ blows to infinity if $A \neq 0$

then $$U(x,s) = \frac{x^2}{s(s+2)} = x^2\left(\frac{K_1}{s} + \frac{K_2}{s+2}\right)$$

$$= x^2\left(\frac{1}{2s} - \frac{1}{2(s+2)}\right)$$

then $$u(x,t) = \mathcal{L}^{-1}\left(U(x,s)\right) = x^2\mathcal{L}^{-1}\left(\frac{1}{2s} - \frac{1}{2(s+2)}\right)$$

$$= x^2 \cdot \frac{1}{2}\left[1 - e^{-2t}\right] = \frac{x^2}{2} - \frac{x^2e^{-2t}}{2}$$

$$\boxed{u(x,t) = \frac{x^2}{2} - \frac{x^2e^{-2t}}{2}}$$

2) (30%)

a) Write down, **without** proof, the modified d'Alembert's formula for the solution of the $1-d$ wave problem:

PDE: $u_{tt} = c^2 u_{xx}$ $\quad 0 < x < \infty, \quad 0 < t < \infty$

BC: $u(0,t) = 0$ $\quad 0 < t < \infty$

ICs $\begin{cases} u(x,0) = f(x) & 0 < x < \infty \\ u_t(x,0) = g(x) & 0 < x < \infty, \quad 0 < t < \infty \end{cases}$

b) Deduce the solution of the IBVP:

PDE: $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{c^2}{x^2}\dfrac{\partial}{\partial x}\left(x^2 \dfrac{\partial u}{\partial x}\right)$ $\quad 0 < x < \infty, \quad 0 < t < \infty$

BC: $u(0,t) = 0$ $\quad 0 < t < \infty$

ICs $\begin{cases} u(x,0) = F(x) & 0 < x < \infty \\ u_t(x,0) = G(x) & 0 < x < \infty, \quad 0 < t < \infty \end{cases}$

Hint: Let $v(x,t) = x.u(x,t)$. Show that $v_{tt} = c^2 v_{xx}$.

a) The modified d'Alembert's formula is:

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi & \text{for } x \geq c \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] - \frac{1}{2c}\int_{ct-x}^{x+ct} g(\xi)\,d\xi & \text{for } x < c \end{cases}$$

b) let $v(x,t) = x\, u(x,t)$

$$\begin{cases} v_t = x\, u_t \\ v_{tt} = x\, u_{tt} \end{cases} \Rightarrow \begin{cases} u_t = \frac{v_t}{x} \\ u_{tt} = \frac{v_{tt}}{x} \end{cases}$$

$$\begin{cases} v_x = u(x,t) + x\, u_x \\ \quad = u + x\, u_x \\ v_{xx} = u_x + u_x + x\, u_{xx} \\ \quad = 2u_x + x\, u_{xx} \end{cases}$$

P.D.E: $\frac{v_{tt}}{x} = \frac{c^2}{x^2}$

$$u(x,t) = \frac{v(x,t)}{x}$$

$$\therefore \quad u = \frac{\cancel{v}}{x}$$

$$u_x = \frac{v_x \cdot x - v}{x^2} \Rightarrow x^2 u_x = v_x x - v$$

then $x^2 u_x = v_x \cdot x - v$

Replacing in the P.D.E

$$u_{tt} = \frac{c^2}{x^2} \frac{\partial}{\partial x}\left[x^2 u_x\right]$$

$$\Rightarrow \frac{v_{tt}}{x} = \frac{c^2}{x^2} \cdot \frac{\partial}{\partial x}\left[v_x \cdot x - v\right]$$

$$\frac{v_{tt}}{x} = \frac{c^2}{x^2}\left[v_{xx} \cdot x + \cancel{v_x} - \cancel{v_x}\right]$$

$$\frac{v_{tt}}{x} = \frac{c^2}{x} v_{xx}$$

$$\Rightarrow \boxed{v_{tt} = c^2 v_{xx}}$$

$$v(0,t) = 0 \cdot u(0,t) = \boxed{0 = v(0,t)}$$

$$\cancel{v}(x,0) = x \cdot u(x,0) = \boxed{x F(x) = v(x,0)}$$

$$v_t(x,t) = x \cdot u_t(x,t)$$

$$v_t(x,0) = x \cdot u_t(x,0) = \boxed{x G(x) = v_t(x,0)}$$

$$\Rightarrow \begin{cases} \text{P.D.E:} & v_{tt} = c^2 v_{xx} \\ \text{BC:} & v(0,t) = 0 \\ \text{ICs:} & \begin{cases} v(x,0) = xF(x) = f(x) \\ v_t(x,0) = x \cdot G(x) = g(x) \end{cases} \end{cases}$$

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from a) we can deduce that the

3) (20%)

a) What is the solution of the vibration string problem:

PDE: $u_{tt} = \alpha^2 u_{xx}$ $\quad 0 < x < L, \quad 0 < t < \infty$

BCs: $\begin{cases} u(0,t) = 0 & 0 < t < \infty \\ u(L,t) = 0 & 0 < t < \infty \end{cases}$

ICs: $\begin{cases} u(x,0) = \sin(2\pi x/L) = f(x) & 0 < x < L \\ u_t(x,0) = (2\pi\alpha/L)\sin(2\pi x/L) = g(x) & 0 < x < L \end{cases}$

b) Graph the solution for $t = 0$ and ~~for $t = 1$.~~

a/ $$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[a_n \sin\frac{n\pi\alpha t}{L} + b_n \cos\frac{n\pi\alpha t}{L}\right]$$

where $$a_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

where $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

thus $$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\cdot \sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow b_n = \begin{cases} \left(\frac{2}{L}\times\frac{L}{2}\right) 1 & \text{for } n=2 \\ 0 & \text{for } n \neq 2 \end{cases}$$

orthogonality property.

then $\boxed{b_2 = 1}$

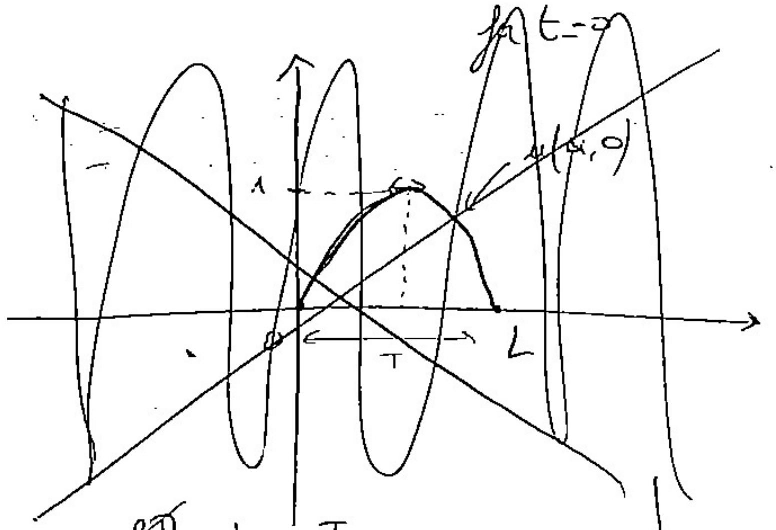
$$a_n = \frac{2}{n\pi\alpha}\int_0^L \left(\frac{2\pi\alpha}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} \frac{2}{2\pi\alpha}\times\frac{L}{2}\times\frac{2\pi\alpha}{L} = & \text{for } n = \\ 0 & \text{for } n \neq \end{cases}$$

from the orthogonality property

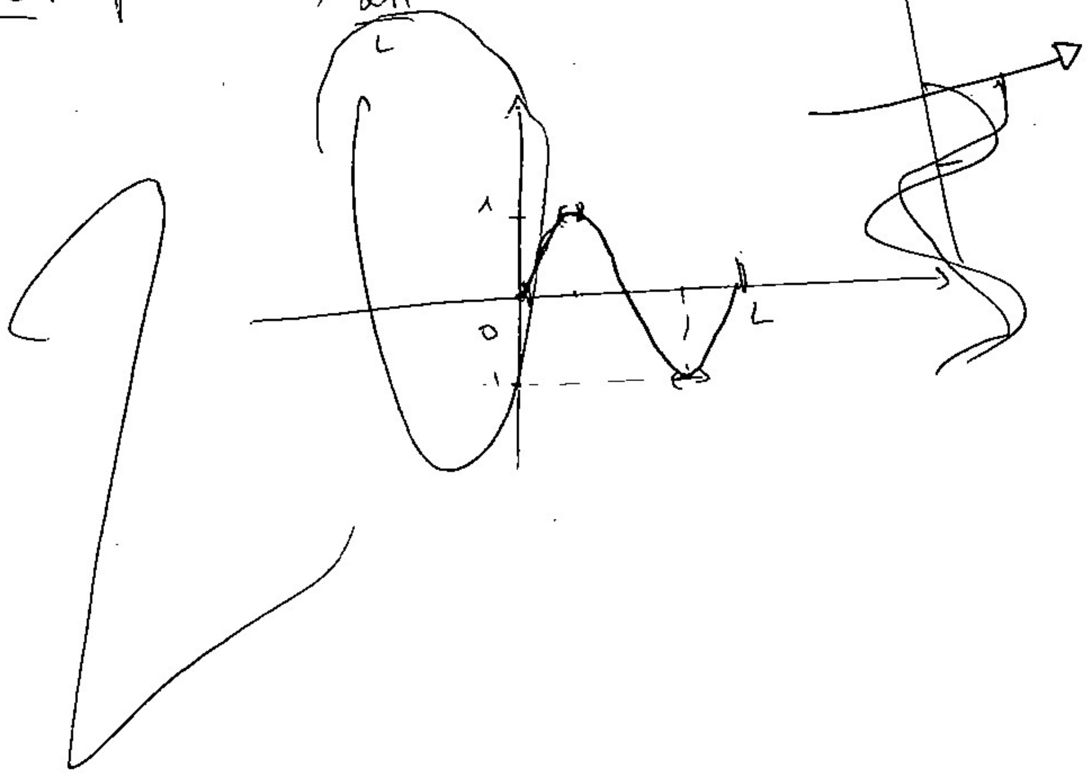
then $u(x,t) = \sin\frac{2\pi x}{L}\left[\sin\frac{2\pi\alpha t}{L} + \cos\frac{2\pi\alpha t}{L}\right]$

b) $u(x,0) = \sin\frac{2\pi x}{L}\left[0+1\right] = \sin\frac{2\pi x}{L}$

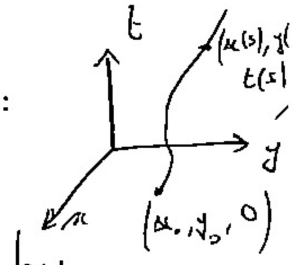
for $t=0$



N.B: period $= \dfrac{2\pi}{\frac{2\pi}{L}} = L = T$



4) **(25%)** Use the method of characteristics to solve the IVP for $u = u(x,y,t)$:
PDE: $u_x + u_y + u_t = 2u \qquad -\infty < x, y < \infty, \qquad 0 < t < \infty$
IC: $u(x,y,0) = f(x,y)$.



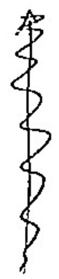
the characteristic = X-curves are given by

$$\begin{cases} \dfrac{dx}{ds} = 1 & \text{with } x(0) = x_0 \\ \dfrac{dy}{ds} = 1 & \text{and } y(0) = y_0 \\ \dfrac{dt}{ds} = 1 & \text{and } t(0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = s + c \overset{I.C}{=} \boxed{s + x_0 = x} \\ y = s + c \overset{I.C}{=} \boxed{s + y_0 = y} \\ t = s + c \overset{I.C}{=} \boxed{s = t} \end{cases}$$

$x - x_0 = y - y_0 = t$

eliminating s $\Rightarrow \begin{cases} x = t + x_0 \\ y = t + y_0 \end{cases} \Rightarrow$ it's the family of the parallel straight lines in the (x, y, t) plane



Along these curves, $u = u(x_0, y_0, s)$ and the P.D.E reduces to

$$\begin{cases} \dfrac{du}{ds} = 2u \\ u(x, y, 0) = f(x_0, y_0) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{du}{ds} - 2u = 0 \\ u(x_0, y_0, 0) = f(x_0, y_0) \end{cases}$$

solving the O.D.E we will get

$$u(x_0, y_0, s) = c e^{2s}$$

$$u(x_0, y_0, 0) = f(x_0, y_0) = c$$

$$\Rightarrow \quad u(x_0, y_0, s) = f(x_0, y_0) e^{2s}$$

then $u(x, y, t) = ??$

$x_0 = x - s$

$y_0 = y - s$ $\Rightarrow \begin{cases} x_0 = x - t \\ y_0 = y - t \end{cases}$

and $s = t$

$$\boxed{u(x, y, t) = f(x - t, y - t) e^{2t}}$$