NDU MAT 335 (PDE) Final Exam Spring 2002 Duration: 2 hours

There are 7 problems total. Do them all.

1(10%) Use the method of separation of variables to solve the nonlinear IVP

PDE: $u_t - 3uu_x = 0$ $\qquad -\infty < x < \infty, 0 < t < \infty$

IC: $u(x,0) = x + 1$ $\qquad -\infty < x < \infty$

2(15%) Solve the nonhomogeneous heat problem

PDE: $u_t = \alpha^2 u_{xx} + \sin 3\pi x$ $\qquad 0 < x < 1, 0 < t < \infty$

BCs: $u(0,t) = u(1,t) = 0$ $\qquad 0 < t < \infty$

IC: $u(x,0) = \sin \pi x$ $\qquad 0 < x < 1$ $\qquad$ (α positive constant)

3(15%) The general solution of the 2nd order PDE $u_{tt} = c^2 u_{xx}$ (c positve constant) is given to be $u(x,t) = \phi(x - ct) + \psi(x + ct)$, where ϕ and ψ are twice differentiable functions of their arguments.

a) Find the solution of the 1-d wave problem

PDE: $u_{tt} = c^2 u_{xx}$ $\qquad 0 < x < \infty, 0 < t < \infty$

BC: $u(0,t) = h(t)$ $\qquad 0 < t < \infty$, ($h(t)$ a given function)

ICs: $\begin{cases} u(x,0) = 0 & 0 < x < \infty \\ u_t(x,0) = 0 & 0 < x < \infty, 0 < t < \infty \end{cases}$

b) Deduce the solution of the 1-d wave problem

PDE: $u_{tt} = c^2 u_{xx}$ $\qquad 0 < x < \infty, 0 < t < \infty$

BC: $u_x(0,t) = h(t)$ $\qquad 0 < t < \infty$, ($h(t)$ a given function)

ICs: $\begin{cases} u(x,0) = 0 & 0 < x < \infty \\ u_t(x,0) = 0 & 0 < x < \infty, 0 < t < \infty \end{cases}$

<u>Hint:</u> Let $v = u_x$ and show that v satiafies the same problem as in (a)

4(15%) Use the method of characteristics to solve the IVP

PDE: $tu_x - xu_t = u$ $\qquad -\infty < x < \infty, \ 0 < t < \infty$

IC: $u(x,0) = f(x)$ $\qquad -\infty < x < \infty$, ($f(x)$ a given function)

What do the characteristics look like? Sketch, in particular, the characteristic (γ) that passes through the point $A(2,0)$

5(15%) The Heaviside unit step function $H(t-a)$ is defined to be

$$H(t-a)=\begin{cases}1 & t\geq a\\ 0 & t<a\end{cases} \quad (a \text{ is constant})$$

a) Find the Laplace Transform of $H(t-a)$

b) Prove that if $F(s)=L(f(t))(s)$, then
$L(f(t-a)H(t-a))(s)=e^{-as}F(s)$

c) Use the Laplace Transform to solve the IBVP

PDE: $xu_t+u_x=x \quad 0<x<\infty,\ 0<t<\infty$
BC: $u(0,t)=0 \quad 0<t<\infty$
IC: $u(x,0)=0 \quad 0<x<\infty$

d) Sketch the solution for $t=1$

6(15%) The solution of Burger's nonlinear IVP

PDE: $u_t+g(u)u_x=0 \quad -\infty<x<\infty,\ 0<t<\infty$
IC: $u(x,0)=\phi(x) \quad -\infty<x<\infty$

is given implicitly by $u=\phi(x-g(u)t)$

Solve the following system

PDEs: $$\begin{cases}u_t-v_t+2u_x=0 & -\infty<x<\infty, 0<t<\infty\\ 2u_t-3v_t+u_x+2v_x=0 & -\infty<x<\infty, 0<t<\infty\end{cases}$$

ICs: $$\begin{cases}u(x,0)=F(x) & -\infty<x<\infty\\ v(x,0)=G(x) & -\infty<x<\infty\end{cases} \quad (F,G \text{ given functions})$$

7(15%) Use the Fourier Transform to show that the solution of the following Dirichlet problem for a half plane

PDE: $\Delta u=u_{xx}+u_{yy}=0 \quad -\infty<x<\infty,\ 0<y<\infty$
BC: $u(x,0)=f(x) \quad -\infty<x<\infty$ (f is a given function)
u is bounded $\quad -\infty<x<\infty,\ 0<y<\infty$

is given by the Poisson integral formula for the half plane

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{yf(\xi)}{(x-\xi)^2+y^2}d\xi$$

Good Luck