Exam 2 - F1

MAT 335 (PDE) Exam # 2 Fall 2004

60 minutes

1. (35%)

a) Verify that $u(x,t) = \phi(x-ct) + \Psi(x+ct)$ is a general solution of the $1-d$ wave equation $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, $0 < t < \infty$.

b) Deduce the solution of the IVP:

PDE: $u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty, \quad 0 < t < \infty$

IC_1: $u(x,0) = f(x)$ $\quad -\infty < x < \infty$

IC_2: $u_t(x,0) = g(x)$ $\quad -\infty < x < \infty, \quad 0 < t < \infty$

c) Graph the solution u versus x for $t = 2$ in the special case where $c = 1$, $f(x) \equiv 0$, and $g(x) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

2. (30%)

a) Write down, <u>without</u> proof, the solution of $1-d$ vibrating string problem:

PDE: $u_{tt} = \alpha^2 u_{xx}$ $\quad 0 < x < L, \quad 0 < t < \infty$

BC_s: $u(0,t) = u(L,t) = 0$ $\quad 0 < t < \infty$

IC_1: $u(x,0) = f(x)$ $\quad 0 < x < L, \quad 0 < t < \infty$

IC_2: $u_t(x,0) = g(x)$ $\quad 0 < x < L, \quad 0 < t < \infty$

b) The ends of a stretched string of length $L = 1$ are fixed at $x = 0$ and $x = 1$. The sting is set to vibrate from rest by releasing it from an initial triangular shape modeled by the function $f(x) = \begin{cases} \frac{3}{10}x & 0 \le x \le 1/3 \\ \frac{3}{20}(1-x) & \frac{1}{3} \le x \le 1 \end{cases}$

Determine the subsequent motion of the string, given that $\alpha = 1/\pi$.

3. (35%) Consider the IVP

PDE: $3u_{xx} + 7u_{xy} + 2u_{yy} = 0$ $\quad 0 < x < \infty, \quad -\infty < y < \infty$

IC_1: $u(0,y) = f(y)$ $\quad -\infty < y < \infty$

IC_2: $u_x(0,y) = g(y)$ $\quad 0 < x < \infty, \quad -\infty < y < \infty$

a) Let $\xi = y - 2x$, $\eta = y - \frac{1}{3}x$. Prove that the above PDE reduces to $\psi_{\xi\eta} = 0$.

b) Solve the given IVP.

Good Luck

PDE EXAM II with Solution

1) (20%) Use D'Alembert's formula to solve the IVP:

$$\begin{array}{lll} \text{PDE: } u_{tt} = c^2 u_{xx} & -\infty < x < \infty, & 0 < t < \infty \\ \text{IC : } u(x,0) = e^{-x^2} & -\infty < x < \infty, & 0 < t < \infty \\ \text{IC : } u_t(x,0) = 0 & -\infty < x < \infty, & 0 < t < \infty \end{array}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi .$$

because $g(x) = 0 \Rightarrow \quad u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] \Rightarrow$

$\cancel{u(x,0) = e^{-x^2} \Rightarrow \frac{1}{2}\left[f(x) + f(x) \right]}$

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2} \right] .$$

the first part is a move to left, the second part is a one to the left.

2) (30%) Use the method of characteristics to solve the IVP:

PDE: $x\,u_x + y\,u_y = 1$ $\quad\quad x>0,\ y>0$

IC : $u = x^2 + y$ $\quad\quad 0 < x = 1 - y < 1$

What do the characteristics look like?

Sketch, in particular, the characteristic that passes through the point $A\left(\frac{1}{2},\frac{1}{2}\right)$.

$$\frac{dx}{ds} = x \Rightarrow x = C_1 e^s \qquad x(0) = x_0$$

$$\frac{dy}{ds} = y \Rightarrow y = C_2 e^s \qquad y(0) = 1 - x_0.$$

$$\Rightarrow \quad C_1 = x_0. \qquad C_2 = 1 - x_0.$$

$$x = x_0 e^s$$

$$y = e^s(1-x_0)$$

$$\left.\begin{array}{l} \frac{du}{ds} = 1 \\ u_{(0,x_0)} = x_0^2 + (1-x_0) \end{array}\right\} \Rightarrow u = s + C. \Rightarrow u_{(s,x_0)} = s + x_0^2 + (1-x_0)$$

$$s = \ln\left(\frac{x}{x_0}\right) \qquad -x_0 = \frac{y}{e^s} - 1 \Rightarrow x_0 = 1 - \frac{y}{e^{\ln(x/x_0)}}$$

$$y = \frac{x}{x_0}(1-x_0) \Rightarrow y x_0 + x x_0 = x \Rightarrow$$

$$x_0 = \frac{x}{y+x}. \qquad s = \ln(y+x) \Rightarrow u(x,y) = \ln(x+y) +$$

$$\left(\frac{x}{y+x}\right)^2 + \left(1 - \frac{x}{y+x}\right)$$

3) **(30%)** Solve the following "Burger" IVP:

PDE: $u_t + u^2 u_x = 0 \qquad -\infty < x < \infty,\ \ 0 < t < \infty$

IC : $u(x,0) = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$

$\phi(g(u(x_0+x_0))) = u$. $\qquad g(u) = u^2$

$x = x_0 + g(u(x_0,0))t$.

if $x \le 0 \Rightarrow$, $x = x_0 + (0 \times t) \Rightarrow x = x_0$.

$\Rightarrow u = 0$ $\qquad$ $g(u(x_0,0))$

if $x > 0 \Rightarrow x = x_0 + x_0^2 t \Rightarrow x_0(1 + x_0 t) = x$

$g(u(x_0,0))$

$u = x_0$

solve for x_0

$\Rightarrow u(x,t) = 0$ if $x \le 0$

$= x_0$ if $x > 0$.

which is the solution of the Burger IVP.

4) **(20%)** Solve the following "Euler" IVP:

PDE: $u_{xx} + 3u_{xy} = 0 \qquad 0 < x \le \infty, \qquad -\infty < y < \infty$

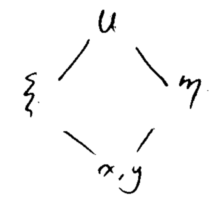
IC : $u(0,y) = f(y) \qquad 0 < x < \infty, \qquad -\infty < y < \infty$

IC : $u_x(0,y) = g(y) \qquad 0 < x < \infty, \qquad -\infty < y < \infty$

Hint: Use the change of variables $\xi = y$, $\eta = y - 3x$.

$\xi_x = 0 \qquad \eta_x = -3$

$\xi_y = 1 \qquad \eta_y = 1$



$x = u_\xi \xi_x + u_\eta \eta_x = -3u_\eta$.

$xx = (u_x)_x = (-3u_\eta)_x = -3(u_\eta)_x = -3[u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x]$.

$= -3[-3u_{\eta\eta}] = 9u_{\eta\eta}$.

$y = -3(u_\eta)_y = -3[u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y] = -3u_{\eta\xi} - 3u_{\eta\eta}$.

place in the PDE equation: $9u_{\eta\eta} - 9u_{\eta\xi} - 9u_{\eta\eta} = 0 \Rightarrow$

$\eta\xi = 0$.

tegration with respect to $\xi \Rightarrow \quad u_\eta = \phi(\eta)$.

ntegration with respect to $\eta \Rightarrow \quad u(\xi,\eta) = \Phi(\eta) + \theta(\xi)$.

$(x,y) = \phi(y-3x) + \theta(y)$.

$(0,y) = \phi(y) + \theta(y) = f(y)$.

$x(x,y) = -3\phi'(y-3x) \Rightarrow \quad u_x(0,y) = -3\phi'(y) = g(y)$

$'(y) = \frac{-1}{3}g(y) \Rightarrow \quad \phi(y) = -\frac{1}{3}\int_{y_0}^{y} g(\psi)\,d\psi$.

$\theta(y) = f(y) - \phi(y)$.

$(x,y) = f(y) - \frac{1}{3}\int^{y} g(\psi)\,d\psi + \frac{1}{3}\int^{y-3x} g\psi\,d\psi$.